ISSN 0081-5438, Proceedings of the Steklov Institute of Mathematics, 2011, Vol. 273, pp. 64–175. © Pleiades Publishing, Ltd., 2011.
Original Russian Text © R.I. Grigorchuk, 2011, published in Trudy Matematicheskogo Instituta imeni V.A. Steklova, 2011, Vol. 273, pp. 72–191.

Some Topics in the Dynamics of Group Actions on Rooted Trees

R. I. Grigorchuk [a,b]

Received May 2010

To the cherished memory of Evgenii Frolovich Mishchenko

Abstract—This article combines the features of a survey and a research paper. It presents a review of some results obtained during the last decade in problems related to the dynamics of branch and self-similar groups on the boundary of a spherically homogeneous rooted tree and to the combinatorics and asymptotic properties of Schreier graphs associated with a group or with its action. Special emphasis is placed on the study of essentially free actions of self-similar groups, which are antipodes to branch actions. At the same time, the theme "free versus nonfree" runs through the paper. Sufficient conditions are obtained for the essential freeness of an action of a self-similar group on the boundary of a tree. Specific examples of such actions are given. Constructions of the associated dynamical system and the Schreier dynamical system generated by a Schreier graph are presented. For groups acting on trees, a trace on the associated C^*-algebra generated by a Koopman representation is introduced, and its rôle in the study of von Neumann factors, the spectral properties of groups, Schreier graphs, and elements of the associated C^*-algebra is demonstrated. The concepts of asymptotic expander and asymptotic Ramanujan graph are introduced, and examples of such graphs are given. Questions related to the notion of the cost of action and the notion of rank gradient are discussed.

DOI: 10.1134/S0081543811040067

1. INTRODUCTION

The modern theory of dynamical systems studies systems defined by group actions, i.e., systems of the form (G, X, μ), where the measure μ is invariant or at least quasi-invariant (semigroup actions are also considered, but this subject is much less developed compared with group actions). The theory also deals with topological dynamical systems of the form (G, X), where X is a topological space and the group G acts by homeomorphisms (topological dynamics). An important class of actions that are considered in modern dynamics is formed by the actions of countable groups, among which a special role is played by the actions of finitely generated groups. The study of rough properties (such as the structure of the partition into orbits) of the actions of countable groups is closely related to the study of countable Borel partitions, while the latter direction is

[a] Department of Mathematics, Texas A&M University, College Station, TX 77843-3368, USA.
[b] Steklov Mathematical Institute, Russian Academy of Sciences, ul. Gubkina 8, Moscow, 119991 Russia.
E-mail address: grigorch@math.tamu.edu

closely linked with modern studies in descriptive set theory (which traces its roots to the Russian scientific school due to the pioneering works of N.N. Luzin and M.Ya. Suslin). The group aspect of dynamical systems theory is also largely due to the Russian mathematical school and is associated with the fundamental studies by N.N. Bogolyubov, I.M. Gel'fand, Yu.V. Linnik, M.L. Gromov, G.A. Margulis, and other outstanding mathematicians. From among the Western school, we should mention, first of all, the studies by J. von Neumann, H. Furstenberg, A. Connes, and R. Zimmer.

In the studies carried out until recently, the (essentially) free actions played a major role, while nonfree actions appeared episodically. Among the first works that dealt with nonfree actions were the studies by the present author [70, 72] and by Vershik and Kerov [185]; the results obtained by the author in the early 1980s were mainly of an algebraic character (related to the geometric and asymptotic directions in group theory) and gave rise to the theory of branch groups and self-similar groups, whereas the studies by Vershik and Kerov were mainly related to representation theory and concentrated around the analysis of the infinite symmetric group $S(\infty)$ and some other locally finite groups. In the last decade, especially after the publications [87, 16], it has become clear that it is important to study group actions on individual orbits for nonfree actions on measure spaces and on topological spaces. This led to the study of Schreier graphs and orbital graphs (associated with actions on orbits). At the same time, two years ago, Vershik put forward a new idea related to the study of the so-called totally nonfree actions. It turned out that the approach of the present author and Vershik's approach have common points; in particular, branch-type actions are totally nonfree (we will touch upon this question below). At the same time, dealing with nonfree actions for many years, I realized the importance of free actions in the case of actions on the boundaries of rooted trees. Therefore, in this paper I pay approximately equal attention to both types of actions and to their relation to various topics.

Originally this paper was planned as a review of a certain range of problems concerning group dynamics on rooted trees, the problems that were first considered about ten years ago in [79, 80, 16, 87, 95]. However, while writing this paper, I came up with new ideas, revealed new relations, and the contours of new directions of investigations started to emerge. Therefore, the paper turned out to be not a pure survey; it contains a lot of new observations and sketches. I devote the following part of the Introduction to the brief description of my philosophy concerning group-action dynamics and then list the contents of the sections of the paper.

There is a close relationship between noncommutative dynamical systems and operator algebras (first of all, von Neumann algebras and C^*-algebras). Any action with a quasi-invariant measure generates a unitary representation of a group; thus, the problems and methods of noncommutative dynamics are often intertwined with the problems and methods of representation theory (which, in turn, are intertwined with the problems and methods of the theory of operator algebras). It is well known that the spectrum of a representation (i.e., its decomposition into irreducible ones) may be either a pure point spectrum (i.e., it may contain only finite-dimensional subrepresentations), a continuous spectrum (i.e., it may contain only infinite-dimensional subrepresentations), or a mixed spectrum (i.e., it may contain both finite-dimensional and infinite-dimensional subrepresentations). For the dynamics of a single automorphism (meaning an action of the cyclic group $\mathbb{Z}$), it is well known that an action with a pure point spectrum is isomorphic to a shift action on a compact abelian group. This classical result by von Neumann and Halmos was generalized by Mackey [131], who proved that faithful ergodic actions with invariant measure and pure point spectrum of a locally compact topological group are isomorphic to the actions of this group on spaces of the form K/H equipped with the image λ of the normalized Haar measure on K, where K is a compact topological group that contains a subgroup isomorphic to the given group or its homomorphic image and H is a closed subgroup of K.

In the classical situation of a single automorphism of a measure space, the discrete case (i.e., the case of a pure point spectrum) is considered trivial (or at least very simple). For the actions of noncommutative groups, the case of a pure point spectrum is not any simpler (and maybe even more complicated) than the case of a continuous spectrum. Of special interest are the actions of discrete groups G on homogeneous spaces K/P of profinite groups (i.e., when K is a compact totally disconnected group). If such an action is faithful, then the group G is embeddable in a profinite group and hence is residually finite (i.e., it has a large family of finite-index subgroups; namely, the intersection of these subgroups is a trivial subgroup). Actually, residually finite groups give precisely the class of groups that have a faithful action with an invariant measure and a pure point spectrum. The Mackey realization of such an action on a homogeneous space K/H may lead to the case when the group K is either connected (and then it is a Lie group), is totally disconnected (the profinite case), or is of mixed type (i.e., it has a nontrivial connected subgroup such that the quotient by this subgroup is totally disconnected). The case of connected K seems to be the simplest case and is the most studied one. All of what is written in the few paragraphs above is well known. Less known are the following facts.

It turns out that dynamical systems of the form $(G, K/P, \lambda)$, $G \leq K$, where K is a profinite group, arise on a seemingly very different basis; namely, they are isomorphic to systems of the form $(G, \partial T, \nu)$, where T is a spherically homogeneous rooted tree, ∂T is its boundary, G acts by tree automorphisms, and ν is a uniform measure on the tree boundary (Theorem 2.9). The first nontrivial actions of this type were considered in [70, 72, 99]; the results obtained there are more related to algebra. The dynamical aspect was given greater attention in [80, 87, 16, 18, 142, 86] and other papers, which initiated a number of new directions of research at the interface between algebra, dynamical systems theory, holomorphic dynamics, theory of operator algebras, discrete mathematics, and other fields of mathematics. Although the theory of actions on trees and their various generalizations (such as $\mathbb{R}$-trees, hyperbolic spaces, buildings, CAT(0)-spaces, etc.) has long become a well-developed theory (an excellent example is given by the material of Serre's book *Trees* [168]), the study of actions on rooted trees has required new concepts and methods and allowed one to reformulate many results related to the widely used group-theory operation of taking the wreath product in geometric and dynamic terms. This fact has made it possible to significantly extend the application domain of this operation (especially under its iteration) and to better understand it. The concept of branch group introduced by the present author [80, 79] is one of the key concepts related to the actions on rooted trees. In terms of dynamical systems, the definition of a branch group looks as follows.

Definition 1.1. (a) A group G acts on a space (X, μ) with invariant measure in a *weakly branch way* if there exists an increasing sequence $\{\xi_n\}_{n=1}^{\infty}$ of finite G-invariant partitions of X that tends to the partition into points and is such that the action of G is transitive on the set of atoms of each partition ξ_n and, for any n and any atom $A \in \xi_n$, there exists an element $g \in G$ acting nontrivially on A and, at the same time, acting trivially on the complement A^c of A.

(b) An action (G, X, μ) belongs to a *branch type* if it is weakly branch and, in addition, for any atom A of any of the partitions ξ_n, the subgroup $\mathrm{rist}_G(A) < G$ consisting of elements that act trivially on the complement A^c has finite index in the restriction $\mathrm{st}_G(A)|_A$ of the stabilizer $\mathrm{st}_G(A)$ of the set A to this set, provided that this subgroup is identified with the restriction $\mathrm{rist}_G(A)|_A$. A group is called a branch group if it has a faithful action of branch type.

Note that the definition of branch groups has never been presented in this form; instead, either a purely algebraic definition or a geometric definition in the language of actions on rooted trees was used [80, 19]. In Section 2, we present a geometric definition and prove that it is equivalent to the one given above.

Branch (just-infinite) groups constitute one of the three subclasses into which the class of just-infinite groups (i.e., infinite groups each proper quotient group of which is finite) is naturally split; it is this fact that primarily determines the importance of branch groups in group theory.

Another important class of groups that act on rooted trees is formed by self-similar groups, in other words, groups generated by Mealy-type automata. Mealy-type automata are automata that operate as transducers, or sequential machines; i.e., these are automata operating as synchronous transducers of information that transmit, letter by letter, an input sequence of letters from a certain alphabet into an output sequence. Invertible initial synchronous automata (more precisely, their equivalence classes) constitute a group with a well-known (in informatics) operation of composition of automata [56, 119]. This group depends on the cardinality of the alphabet; i.e., in fact, there exists a sequence of groups that is indexed by positive integers (by the cardinality of the alphabet).

If we consider a more general class of automata, namely, the asynchronous automata, then, as shown in [87], there is only one universal group, independent of the cardinality of the alphabet, in which all groups of synchronous automata are embedded. In [87] this group was called the group of rational homeomorphisms of the Cantor set. In addition to self-similar groups, it contains other quite interesting subgroups, for example, the famous Thompson groups. A group is said to be self-similar if it is isomorphic to a group generated by the states of a noninitial invertible synchronous automaton.

The groups generated by finite automata (we call them strongly self-similar groups in this paper) are specially distinguished. A simple example of a self-similar group is the infinite cyclic group, which can be realized by the action of an odometer (also called an *adding machine* in the English-language literature and often translated into Russian as a *d-adic counter*, where d is the cardinality of the alphabet). The odometer acts in the space of right-infinite sequences of letters in an alphabet of cardinality $d \geq 2$ equipped with a uniform Bernoulli measure, or, equivalently, on the boundary of a d-regular rooted tree (there is a generalization of the concept of odometer to the case when the phase space is the Cartesian product of a sequence of various alphabets). This dynamical system with discrete spectrum is well known in ergodic theory. A considerably more complex example of a (strongly) self-similar group is given by a group $\mathcal{G}$ that was constructed by the author in [70] and then studied in [72] and many other papers. The main properties of this group are the periodicity, intermediate growth (between polynomial and exponential), and nonelementary amenability.

Self-similar groups, especially those possessing the branch property, form quite an interesting class of groups related to many aspects of dynamical systems theory and other fields of mathematics. The theory of iterated monodromy groups developed by Nekrashevych [142] has breathed new life into holomorphic dynamics and found wide applications in the study of Julia sets and other fractal objects [142].

Actions on rooted trees turned out to be useful for the theory of profinite groups, since any profinite group with a countable base of open sets is embedded in the automorphism group (equipped with the natural topology) of an appropriate rooted tree T. Moreover, if a group G acts transitively on the levels of a tree (or, equivalently, its action on the boundary is ergodic), then the closure $\overline{G}$ in $\mathrm{Aut}(T)$, which is a profinite group, acts transitively on the boundary ∂T, and the uniform measure ν becomes the image of the Haar measure on $\overline{G}$. In this case, the dynamical system $(G, \partial T, \nu)$ is isomorphic to the system $(G, \overline{G}/P, \nu)$, where $P = \mathrm{st}_{\overline{G}}(\xi)$, $\xi \in \partial T$. As already mentioned, the converse is also true; namely, any action with pure point spectrum of type $(G, K/P, \mu)$, where K is a profinite group, is isomorphic to the action on the boundary of an appropriate rooted tree (which can easily be verified by applying the construction of the action of a residually finite group on a coset tree as described in the next section; see Theorem 2.9 there). Thus, in the Mackey theorem, the profinite case corresponds to actions on rooted trees. Another argument in favor of rooted trees is that, as noticed in [87], any compact homogeneous ultrametric space is isometric to the boundary of a rooted tree with an appropriate metric on it (a weaker version of this statement is contained in [59]).

When studying group actions, one usually assumes that the actions are essentially free, i.e., for any nonidentity element of a group, the measure of the fixed point set is zero. One of the first attempts to draw attention to the case of actions that are not essentially free was made in [70] and the following studies [72, 73, 99], which led to the concept of branch action and, accordingly, branch group. Obviously, a weakly branch action is not essentially free. Paper [185] by Vershik and Kerov was also one of the pioneering works on the use of actions that are not essentially free. The importance of studying nonfree actions has recently been pointed out by Vershik in [184]. The theme of the "free versus nonfree action" alternative runs through the larger part of our paper.

An important object that arises when studying actions that are not essentially free is an orbital graph Γ_ξ, $\xi \in X$, of an action (on the space X). The vertices of this graph are points of the orbit and the edges correspond to transitions from one vertex to another under the action of a generator (in this case, the edges are labeled by the respective generators). If the action is essentially free, then such graphs are almost surely isomorphic to the Cayley graph constructed for the group by means of the same system of generators. For actions that are not essentially free, the graphs Γ_ξ are almost surely nonisomorphic to the Cayley graphs but are isomorphic to Schreier graphs, i.e., to graphs of the form $\Gamma = \Gamma(G, H, A)$, where $H \leq G$ is a subgroup (corresponding to the stabilizer of some boundary point) and A is a system of generators. The vertices of such a graph are left (one may also consider right) cosets gH, and two vertices fH and gH are connected by an oriented edge labeled by a generator $a \in A$ if $gH = afH$. The Cayley graphs are obtained in this construction when H is the trivial subgroup. Depending on the situation, one can consider various modifications of the concept of a Schreier graph: one can make edges nonoriented, remove labels from them, distinguish a vertex in a graph and consider it as a root, etc. According to the category chosen, it is expedient to consider appropriate spaces of graphs with natural compact topology and speak of the convergence of graphs in this topology. For example, a topology in the space of Cayley graphs was first defined in [72] and used for studying group properties such as intermediate growth, impossibility of presentations by a finite set of relations, Kolmogorov complexity of the word problem [74], etc. Later, this topology and its variations were examined more carefully (first of all in [43]), and now it plays a significant role in many investigations. Note that, in the much earlier work [42], Chabauty introduced a topology on the set of closed subgroups of a locally compact group and applied it to the study of lattices in such groups. The Chabauty topology is widely used in the studies of lattices in Lie groups (see [158]). In terms of this topology one can also interpret topologies in the spaces of Cayley graphs and Schreier graphs.

The first publications in which the authors realized the importance of studying Schreier graphs that arise as orbital graphs of actions are [16, 87]. For example, the following simple but important fact is borrowed from [87, Proposition 6.22].

Proposition 1.1 [87]. *Let G be a finitely generated group that acts ergodically on a space (X, μ) by transformations that preserve the measure μ (i.e., the measure μ is quasi-invariant). Then the Schreier graphs of the action on orbits are μ-almost surely locally isomorphic to each other.*

In this proposition the local isomorphism of two graphs means that, for any radius r and an arbitrary vertex of any of the graphs, there exists a vertex of the other graph such that the neighborhoods (subgraphs) of radius r around the chosen vertices in the two graphs are isomorphic. A similar proposition is valid in the topological situation as well, but it requires the concept of a G-typical point and a slight correction in the formulation of the proposition above; moreover, there are examples of graphs that are generic in the topological sense but are not generic in the metric sense [2].

Schreier graphs associated with the actions of self-similar groups and branch-type groups on the levels and the boundary of a tree are important both for solving various problems of graph theory and for studying asymptotic problems involving graphs and groups. These graphs model various phenomena and reflect the complexity of many related problems. For example, the classical Tower of Hanoi problem with four or more pegs is equivalent to calculating the distance between specific vertices in these graphs and finding the shortest path between them in an algorithmic manner.

See [91–93] for more details on this subject. There are a lot of questions that arise when studying Schreier graphs; first of all, these questions are related to group theory and dynamical systems. These are questions on the number of ends of the graphs, on the growth, amenability, and the possibility to define the graph by a finite system of substitution rules, on the possibility of reconstructing a system from a generic Schreier graph, on the asymptotic behavior of the first nonzero eigenvalue of the discrete Laplace operator, on the spectrum of the discrete Laplace operator, on the construction of expanders, on the asymptotic behavior of random walks, on the calculation of the cost of actions and cost of groups, etc. Many of these questions are touched upon in the present paper or in the references cited. One of the new results given below is the construction of asymptotic expanders on the basis of finite automata (and on the basis of related self-similar groups). It is an interesting open problem to find out whether these graphs are true expanders. Another circle of questions related to the study of actions on rooted trees is the study of infinite decreasing chains of finite-index subgroups in residually finite groups, in particular, the study of the rank gradient of these subgroups [121, 5].

As already mentioned, actions on rooted trees and the problems of self-similar groups are mysteriously related to many questions of dynamical systems theory. These questions arise when restricting the actions to Lyapunov stable attractors [87, Theorem 6.16]. They are related to substitution dynamical systems (which arise when finding presentations of groups by generators and relations) [129, 97, 14]. The description of invariant subsets of multidimensional rational mappings served as a basis for a new unexpected method of solution of the spectral problem for the discrete Laplace operator [16]. Owing first of all to the studies of Nekrashevych, the theory of iterated monodromy groups has led to significant changes in the strategy of studies on holomorphic dynamics [142].

In Section 8, we propose a construction of a Schreier dynamical system: given a combinatorial structure (a Schreier graph) or algebraic data (a pair consisting of a group and its subgroup), one can use this construction to obtain a dynamical system. The examples presented in Section 8 show that in many cases the original dynamical system can be recovered from this construction if one takes the orbital graph of the action on a specific orbit as the Schreier graph, or if one considers the stabilizer of a point of the phase space as a subgroup of the acting group.

In essence, all new results concerning the structure of the class of amenable groups, which was introduced by von Neumann and independently by Bogolyubov, as well as the class of intermediate growth groups (about which Milnor asked whether it is empty) have been obtained on the basis of studying group actions on rooted trees. An original method for proving the amenability, called a "Münchhausen trick," was developed in [23, 109]. Various operator algebras associated with actions on rooted trees (as well as with Cuntz algebras in some cases) were defined and studied in [16, 141, 86]. It turned out that among these algebras there are both simple C^*-algebras and algebras that can be approximated by finite-dimensional algebras, similar to residually finite groups. The classical method known as the "Schur complement" found an unexpected application to these algebras in [86]. This list of research directions related to actions on rooted trees is far from complete but we stop here.

Now we briefly outline the contents of the paper. Section 2 is of preliminary character and contains many definitions used in the paper. First of all, we define the main concepts related to spherically homogeneous rooted trees and actions on them. We give a different (compared with Definition 1.1) definition of branch groups and explain how to construct a rooted tree by a decreasing chain of subgroups of finite index. We define just-infinite groups and hereditary just-infinite groups and formulate a theorem describing the trichotomy of the structure of the class of just-infinite groups. We give examples of groups and of their actions.

Section 3 is devoted to groups of automata and self-similar groups. We explain what wreath recursions are. We give definitions of a contracting group and a self-replicating group.

Section 4 is devoted to studying essentially free actions on the boundary of a tree. Abért and Virág [6] proved that a randomly chosen action of a group with a finite number of generators on a binary tree is free (moreover, the group itself is free). However, the explicit construction of an essentially free action is, as a rule, a complicated problem. We present a number of conditions of algebraic character that guarantee the freeness of an action and discuss the relationship between the freeness of actions in the topological and metric senses. Although in the general case there is no direct relation between topological freeness and freeness in the metric sense (i.e., with respect to a measure), a remarkable fact is that for the actions of strongly self-similar groups the two concepts are equivalent; this result was obtained by Kambites, Silva, and Steinberg [112].

In Section 5, we give specific examples of essentially free actions. We consider both the actions of well-known groups, such as the lamplighter realized by a two-state automaton [95], and new actions, and discuss an approach to finding out under what conditions a self-similar group acts essentially freely. A certain role in this discussion is played by the Mikhailova subgroups of the direct product of two copies of a free group.

In Section 6, we consider various topologies on the spaces of Schreier graphs and prove the Gross theorem stating that any connected regular graph with even-degree vertices can be realized as a Schreier graph of a free group.

In Section 7, we give examples of Schreier graphs related to self-similar groups. We define various types of substitution rules and recursions for infinite sequences of finite graphs. The main objects here are the Schreier graphs of the group $\mathcal{G}$ of intermediate growth and of the group called the Basilica. The material of this section is mainly based on the publications [16, 142, 93, 31, 81, 48].

In Section 8, we describe a construction that starts with a dynamical system and yields an associated dynamical system in the space of Schreier graphs or in the space of subgroups of a group. This material correlates with some questions touched upon in [184]. In addition, we describe a technique that allows one to construct an action of a group on a certain compact set by an infinite Schreier graph of this group. This technique leads to interesting actions when the automorphism group of the graph is small (for example, trivial). We show how this technique works in the case of the group $\mathcal{G}$ and the Thompson group (for the latter we use the results of Vorobets [188] and Savchuk [166]). An interesting fact exhibited in these examples is that on the metric level a dynamical system is reconstructed by the Schreier graph, whereas on the topological level the arising space and action are simple perturbations of the original phase space and an action on it. In general, the approach proposed in this part of the paper to the study of dynamical systems should have been called an *orbit method* in the dynamics of finitely generated groups, by analogy with Kirillov's orbit method in representation theory. We stress that while Kirillov's orbit method is mainly used in the representation theory of Lie groups, our approach applies to actions of countable groups equipped with a discrete topology. In this section, we also emphasize the importance of weakly maximal subgroups for the orbital approach to dynamical systems and present some results from [16, 17] (most of which are known) that are related to this class of subgroups. We also present E. Pervova's nontrivial example of a weakly maximal subgroup of the intermediate growth group $\mathcal{G}$.

In Section 9, we discuss unitary representations of groups acting on trees and consider C^*-algebras associated with these representations (we also touch upon von Neumann algebras). On one of these C^*-algebras, we define a trace, which is called a recurrent (or self-similar) trace, and describe some of its properties. Here we mainly use the results obtained in [16, 95, 141, 86, 184]. The recurrent trace has additional useful properties in the case when a group is strongly self-similar. For the intermediate growth group $\mathcal{G}$, we give an explicit description of the values of the trace on the elements of the group. We discuss some properties of the C^*-algebras under consideration. We show that the weakly branch groups belong to the class of ICC (infinite conjugacy class) groups, which possess infinite (nontrivial) conjugacy classes of elements.

In Section 10, we consider questions related to random walks on groups and graphs, the spectral properties of the discrete Laplace operator (or, equivalently, of the Markov operator related to a random walk), as well as the Kesten spectral measure and the so-called KNS (Kesten–von Neumann–Serre) spectral measure, which was introduced and examined in [16, 98]. Examples of a self-similar essentially free action of a free rank 3 group and of the free product of three copies of an order 2 group and results on the recurrent trace are used for constructing asymptotic expanders. We discuss various questions concerning the asymptotic behavior of infinite graphs and infinite covering sequences of finite graphs.

Finally, in Section 11 we discuss questions related to the concept of the cost of actions of countable groups and of countable Borel equivalence relations, as well as the concept of rank gradient of infinite decreasing sequences of finite-index subgroups. This material is based on the studies by G. Levitt, D. Gaboriau, M. Lackenby, M. Abért, and N. Nikolov. We discuss the problems of amenability and hyperfiniteness of groups and equivalence relations and present classical results associated with the names of H. Dye, J. Feldman, C. Moore, A. Connes, and B. Weiss. We introduce the concepts of self-similar and self-replicating equivalence relations and show that the latter are "cheap" in the sense of cost.

This paper is mainly a survey that summarizes the results of research carried out during the last decade in a certain direction. However, it also presents some new observations. Moreover, the paper formulates many open questions. I hope that this paper will stimulate further investigations in the field of dynamics with a pure point spectrum, dynamics of actions on trees, and other related fields of mathematics.

Since the paper is addressed to readers involved in different fields of mathematics, starting from algebraists and ending with specialists (or beginners) in dynamical systems theory, theory of operator algebras, and discrete mathematics, in many places I go into greater detail than I should have to if the paper was addressed only to the reader involved in one specific field. Sometimes, I do not consider it beneath me to remind an already introduced notion or an already formulated result. I hope that the reader will not judge me harshly for this.

2. ACTIONS ON ROOTED TREES

Let $\overline{m} = \{m_n\}_{n=1}^{\infty}$, $m_n \geq 2$, be a sequence of positive integers (called a branch index in what follows) and $T_{\overline{m}}$ be a spherically homogeneous rooted tree defined by the sequence $\overline{m}$. This tree has a root vertex denoted by $\varnothing$, m_1 vertices of the first level, $m_1 m_2$ vertices of the second level, and generally $m_1 m_2 \dots m_n$ vertices of the nth level, $n = 1, 2, \dots$. Each vertex of level n has m_{n+1} "successors" situated at the next level and connected by an edge with this vertex. A clear idea of a rooted tree is given by Fig. 2.1. Note that according to the tradition established in Russian mathematics, a tree is depicted top down.

The norm of a vertex u (denoted by $|u|$) is the level to which this vertex belongs. When the sequence $\overline{m}$ is constant, i.e., $m_n = d$ for some $d \geq 2$ and any n, the tree $T_{\overline{m}}$ is called a *regular*

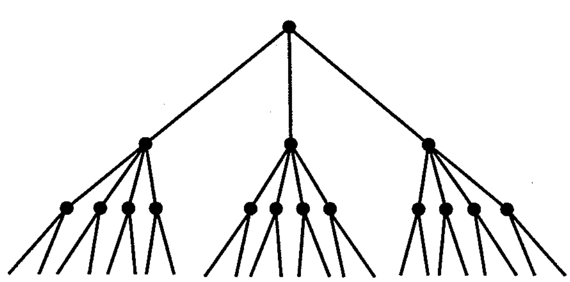

Fig. 2.1. A spherically homogeneous rooted tree.