Amenable Actions of Nonamenable Groups

Rostislav Grigorchuk* and Volodymyr Nekrashevych

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1 Introduction

Since 1929 when von Neumann [vN29] introduced the notion of an invariant mean on a group (and more generally on a G-set) there is a permanent interest in the study of the phenomenon known as amenability. Amenable objects like groups, semigroups, algebras, graphs, metric spaces, operator algebras etc. play an important role in different areas of mathematics. A big progress in understanding of the structure of the class of amenable groups and in the study of asymptotic characteristics of them like growth of Følner sets (the notion introduced by A.M. Vershik in [Ver73]), drift, entropy etc. was reached in the past two decades [Ver73, Gri85, KV83, Gri98, CSGH99, BV, Ers04, Ers03, Ers05, BKNV04].

An important role in propaganda of the idea of amenability belongs to, perhaps the best, introductory to the subject of amenable groups book of Greenleaf [Gre69] where the following question is formulated.

Q1. Let X be a G-set and there is an invariant mean for the pair (G, X). Does this imply that the group G is amenable?

Here one has to add some extra conditions in order to avoid immediate negative answer to the question. Namely, one has to assume that the group G acts faithfully (otherwise the pair $(F_m, F_m/N)$ would be a trivial counterexample where F_m is a free group of rank $m \geq 2$, $N \triangleleft F_m$ is a normal subgroup such that the quotient F_m/N is amenable and F_m acts on F_m/N in the standard way). The second reasonable assumption is transitivity of the action of G on X. Otherwise one can take X equal to a union of G-orbits and then existence of an invariant mean for (G, X) would follow from the existence of an invariant mean for any pair (G, Gx), where $x \in X$. Of course the action of G on orbits can be nonfaithful even in case it is faithful on X, but certainly a transitive amenable pair (G, X) with nonamenable G can be viewed as a more interesting example giving the negative solution of the above question. So we reformulate the Greenleaf question as follows.

Q2. Let a group G act transitively and faithfully on a set X. If the pair (G, X) is amenable (i.e. there is G-invariant mean on X) does this imply amenability of G?

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Surprisingly, the question of Greenleaf did not attract attention of a large community of mathematicians, although it was solved (in negative) in [vD90]. But recently the interest to this question came back and a number of new constructions are on the way to print. This is stimulated, in particular, by the observation made in [MP] that coamenability of subgroups (a subgroup $H < G$ is coamenable if a pair $(G, G/H)$ is amenable) behaves differently from coamenability of normal subgroups.

We are going to present in this note two constructions of amenable actions of nonamenable groups. In the first construction G is a free noncommutative group and the action (G, X) is viewed as a Schreier graph of G, so that amenability of the action is converted to amenability of the graph. A similar construction appears in [GKN05].

In the second example we use the methods of the theory of groups acting on rooted trees developed in [Sid98, Gri00, BGŠ03, GNS00]. We start with an arbitrary nonamenable residually finite group G, realize it as a group acting on a spherically homogeneous rooted tree T and then extend it to a group $\widetilde{G}$ (also acting on a rooted tree) in a way, which guarantees amenability of the pair $(\widetilde{G}, \widetilde{G}/P)$ where P is a parabolic subgroup (i.e. the stabilizer of a point of the boundary ∂T).

The next question naturally arises as a part of our investigation.

P1. For which nonamenable groups G there is a faithful transitive and amenable action (G, X)? (i.e. there is a coamenable subgroup $H < G$ with the core

$$\bigcap_{g \in G} g^{-1} H g$$

being trivial). Let us call such groups NAA groups. Observe that groups with Kazhdan T-property are not NAA groups.

P2. Is there a finitely generated nonamenable group without property (T) and without NAA property?

As far as we know, Y. Glasner and N. Monod have other two constructions of amenable pairs (G, X) with nonamenable G. It would be interesting to get more on such constructions.

2 The first construction

In our first construction the group G will is the free group F_m of rank $m \geq 2$, the set X is the set F_m/H of cosets gH, $g \in F_m$, where $H < F_m$ is a coamenable subgroup with trivial core (so that the left action $(F_m, F_m/H)$ is amenable, faithful and transitive).

The group H will be constructed in a combinatorial-geometric way via the construction of a $2m$-regular amenable graph (with some extra properties) which will be converted to a Schreier graph $\Gamma = \Gamma(F_m, H, S)$ where $S = \{a_1, \ldots, a_m\}$ is a free set of generators of F_m.

Remind that the set of vertices of the graph Γ is identified with the set of left cosets gH, $g \in F_m$ and two "vertices" gH and hH are connected by an oriented edge labelled by s if $gH = shH$, where $s \in S \cup S^{-1}$. Obviously the degree of each vertex of this graph is $2m$. Amenability of the pair

(F_m, H) is equivalent to amenability of the graph Γ, which can be defined as existence of a sequence $\{F_n\}$ of finite subsets of Γ with the property that $|\partial F_n|/|F_n| \to 0$ as $n \to \infty$, where ∂F_n is the boundary of F_n (for amenability of graphs see [CSGH99]). One of properties that insure amenability of a graph is subexponentiality of the growth [CSGH99] (which means that the number of vertices in Γ of combinatorial distance $\leq n$ from a distinguished vertex v_0 grows slower than exponential functions). It is known [Har00] that every $2m$-regular (nonoriented and without labelling of edges) graph Δ can be converted to a Schreier graph of a free group F_m by putting an orientation on the edges and labelling of the edges by the elements of the set $S \cup S^{-1}$. Therefore any example of a $2m$-regular graph of subexponential growth leads to a construction of an amenable pair $(F_m, F_m/H)$. A free generating set of the subgroup $H < F_m$ can be found in the following way.

Construct a spanning subtree T in Δ and let E_0 be the set of the edges of Δ that do not belong to T. For each $e \in E_0$ let t_e be the path peq where p is the geodesic path in T joining the initial vertex v_0 of Δ with the beginning of the edge e and q is the geodesic path in T joining the endpoint of e with v_0. Let w_e be the word read along the path t_e. Then $\{w_e, e \in E_0\}$ is a free set of generators of H.

There are plenty of $2m$-regular graphs of subexponential (even polynomial) growth. The problem only is in getting the faithfulness of the action of F_m on F_m/H, i.e., in showing that the core

$$\bigcap_{g \in F_m} g^{-1} H g \tag{1}$$

is trivial. The last step in our construction is to show how to construct the graph Δ which guarantees the triviality of the core (1).

A word w over the alphabet $S \cup S^{-1}$ represents an element of H if and only if the path l_w in Δ starting in the vertex v_0 and determined by the word w is closed. If we change the reference vertex v_0 by a vertex u_0, then we will replace the group H by its conjugate $g^{-1}Hg$, where g is an element given by a word that can be read on any path joining v_0 with u_0. Thus, if we construct a graph Δ with the property that for any nonempty freely reduced word w there is a vertex u of the graph such that the path beginning in u and determined by the word w is not closed, then the core of H will be trivial. This property is satisfied if for any positive integer r there is a vertex u_r of the graph, such that the length of any back-trackless loop in Δ beginning in u_r is greater than r (i.e. the neighborhood of u_r in Δ of radius r is a tree).

Construction of a $2m$-regular graph which satisfies all the listed properties is easy. Start with an m-dimensional grid $\mathbb{Z}^m = \Delta_0$, where $m > 1$, and make a sequence of local surgeries in it by replacement at the r-th step the 1-neighborhood of a vertex u_r of Δ_0 (see Figure 2) by the graph shown next on Figure 2, where Γ_r is any $(2m-1)$-regular graph with $2m(2m-1)^{r-1}$ vertices.

The graph Δ_0 has polynomial growth of degree $2m$. It is clear that if we choose the sequence $\{u_r\}_{r=1}^{\infty}$ of vertices in Δ_0 such that the distance d_r of u_r from the origin 0 of $\Delta_0 = \mathbb{Z}^m$ is growing very fast than the graph Δ obtained from Δ_0 by such local reconstructions will have a polynomial growth (and hence will be amenable) and the core of the corresponding group H will be trivial.

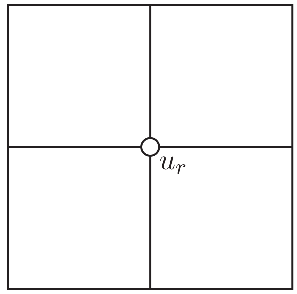

Figure 1:

3 Automorphism groups of rooted trees

Let $\mathsf{X} = (X_1, X_2, \ldots)$ be a sequence of finite sets and let us denote by X^* the set of words $x_1 x_2 \ldots x_n$, where $x_i \in X_i$, together with the empty word $\varnothing$. Let us denote

$$\mathsf{X}^n = \{x_1 \ldots x_n \ : \ x_i \in X_i\} = X_1 \times \cdots \times X_n$$

and $\mathsf{X}^0 = \{\varnothing\}$. Then $\mathsf{X}^* = \bigsqcup_{n \geq 0} \mathsf{X}^n$.

We can transform X^* into a rooted tree in a natural way: connect every vertex $v \in \mathsf{X}^n$ to the vertices of the form vx for $x \in X_{n+1}$. The empty word $\varnothing$ is the root of the tree. The tree X^* is called the *spherically-homogeneous tree* of the spherical index $(|X_1|, |X_2|, \ldots)$. The spherical index determines the tree uniquely, up to an isomorphism of rooted trees.

We denote $\mathsf{X}_n = (X_{n+1}, X_{n+2}, \ldots)$. The spherically homogeneous tree X^* is called *regular* if its spherical index is constant. In this case we may assume that the sequence $\mathsf{X} = (X_1, X_2, \ldots)$ is also constant. In this case $\mathsf{X}_n = \mathsf{X} = (X, X, \ldots)$.

The *boundary* X^ω of the tree X^* is identified with the set of the infinite words of the form $x_1 x_2 \ldots$, where $x_i \in X_i$. The disjoint union $\mathsf{X}^\omega \sqcup \mathsf{X}^*$ has a natural topology defined by the base consisting of the *cylindrical sets*

$$v\mathsf{X}^\omega_{|v|} \sqcup v\mathsf{X}^*_{|v|}$$

of words starting with a given finite word v. Here $|v|$ denotes the length of the word v, i.e., $v \in \mathsf{X}^{|v|}$. The topological space $\mathsf{X}^\omega \sqcup \mathsf{X}^*$ is compact and totally disconnected. The subspace X^ω is homeomorphic to the Cantor set and the topology on it is the direct product topology of the finite discrete sets X_i. The subset X^* is discrete and dense in $\mathsf{X}^\omega \sqcup \mathsf{X}^*$.

The boundary X^ω also has a natural measure (that we call *Bernoulli measure*) equal to the direct product of the uniform probability measures on the sets X_i. It is the unique measure invariant under the action of the full automorphism group of X^*.

We are interested in groups acting faithfully on the tree X^* by automorphisms. Every such an action extends in a unique way to an action by homeomorphisms on $\mathsf{X}^\omega \sqcup \mathsf{X}^*$. The obtained action on X^ω is measure-preserving.

An action of G on X^* is said to be *level-transitive*, if it is transitive on every level X^n of the tree X^*. The action is level-transitive if and only if the induced action on X^ω is minimal, i.e., has dense

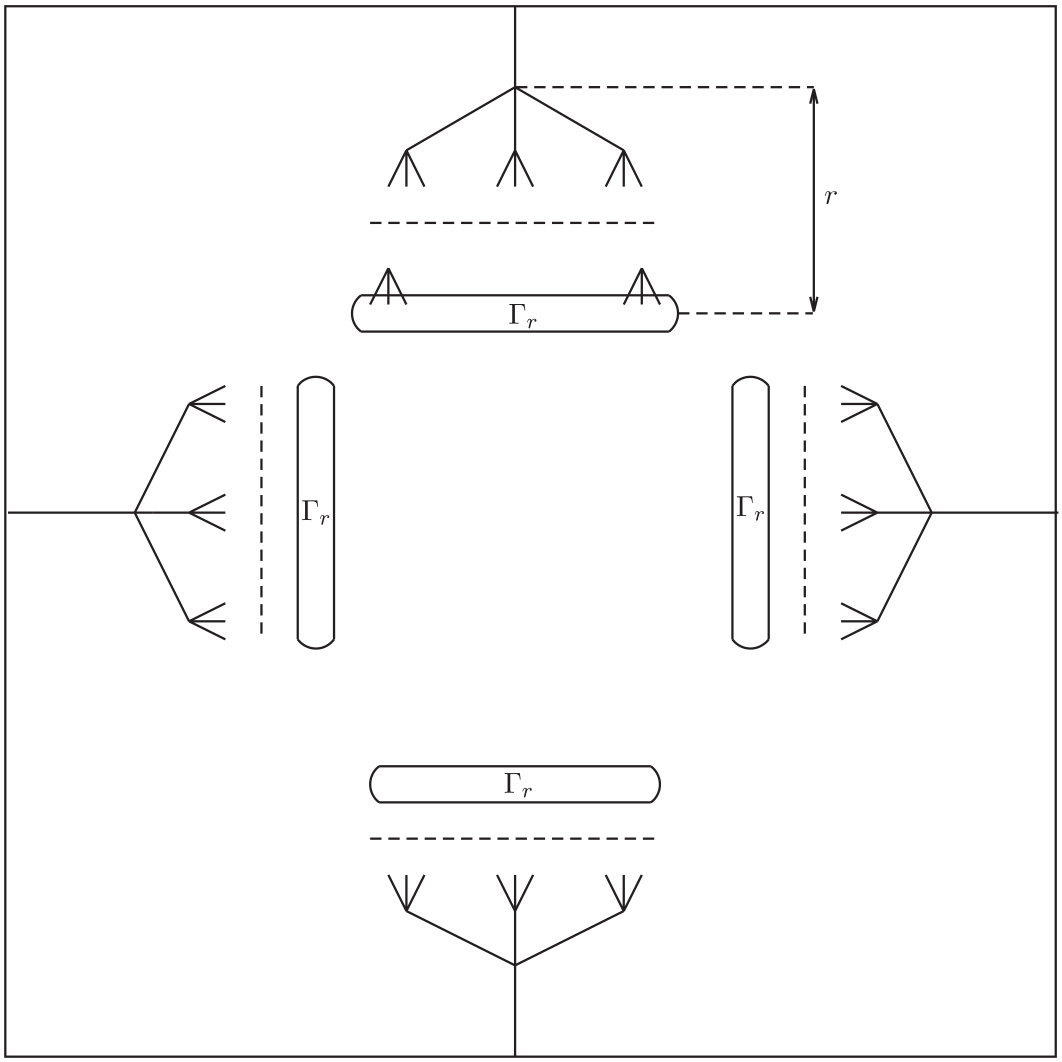

Figure 2:

orbits (see [GNS00]). In particular, if an action of G on X^* is faithful and level-transitive, then the restriction of the action onto any G-orbit of X^ω is also faithful.

An action is level-transitive if and only if it is ergodic with respect to the Bernoulli measure on X^ω.

If g is an automorphism of X^* and $v \in \mathsf{X}^n$ is a word, then the *restriction* $g|_v$ is the automorphism of the tree X_n^* defined by the condition that

$$(vw)^h = v^h w^{g|_v}$$

for all $w \in \mathsf{X}_n^*$. It is easy to see that $g|_v$ is uniquely defined and is an automorphism of X_n^*.

If the sequence X is constant (and hence $\mathsf{X}_n = \mathsf{X}$ for all n), then an automorphism g of X^* is said to be *finite-state* if the set $\{g|_v \ : \ v \in \mathsf{X}^*\}$ is finite. The set of all finite-state automorphisms is a countable subgroup of the automorphism group of X^*, is called the *group of finite automata* and is denoted $\mathcal{F}(\mathsf{X})$.

If g is an automorphism of the tree X^*, then we say that an infinite word $x_1x_2\ldots \in \mathsf{X}^\omega$ is *g-rigid*, if there exists such n that $g|_{x_1\ldots x_n}$ is trivial. The set of g-rigid points is obviously open.

We say that an automorphism g of X^* is *almost finitary*, if the set of g-rigid points of X^ω has full measure.

Proposition. *The set of almost finitary automorphisms of X^* is a group.*

Proof. We have the following obvious properties of restrictions

$$g|_{v_1v_2} = g|_{v_1}|_{v_2}, \quad (g_1g_2)\,|_v = (g_1|_v)\,(g_2|_{v^{g_1}})\,, \quad (g^{-1})\,|_v = \left(g|_{v^{g^{-1}}}\right)^{-1}.$$

Suppose that g is almost finitary. Then for almost every sequence $x_1x_2\ldots \in \mathsf{X}^\omega$ there exists n such that $g|_{(x_1\ldots x_n)^{g^{-1}}} = 1$, since g^{-1} is measure-preserving. But then

$$g^{-1}|_{x_1\ldots x_n} = \left(g|_{(x_1\ldots x_n)^{g^{-1}}}\right)^{-1} = 1,$$

which proves that g^{-1} is almost finitary.

Suppose now that g_1, g_2 are almost finitary. Then for almost every $x_1x_2\ldots \in \mathsf{X}^\omega$ there exists n such that $g_1|_{x_1\ldots x_n} = 1$ and $g_2|_{(x_1\ldots x_n)^{g_1}} = 1$, since g_1 is measure-preserving. Then we have

$$(g_1g_2)\,|_{x_1\ldots x_n} = g_1|_{x_1\ldots x_n} \cdot g_2|_{(x_1\ldots x_n)_1^g} = 1,$$

i.e., g_1g_2 is also almost finitary. □

Let us denote by $\mathcal{A}(\mathsf{X})$ the group of almost finitary automorphisms of X^*. We have the following examples of almost finitary automorphisms.

Proposition. *Let X be constant. An element $g \in \mathcal{F}(\mathsf{X})$ is almost finitary if for every $v \in \mathsf{X}^*$ there exists $u \in \mathsf{X}^*$ such that $g|_{vu} = 1$.*

Proof. Let $\{g_1, g_2, \ldots, g_n\} = \{g|_v : v \in X^*\}$ be the set of the states of g. There exists v_1 such that $g_1|_{v_1} = 1$. There exists v_2 such that $g_2|_{v_1 v_2} = 1$, and further, by induction, there exists a sequence $v_1, v_2, \ldots, v_n$ of words such that $g_i|_{v_1 v_2 \ldots v_i} = 1$ for $i = 1, \ldots, n$. Then for every $i = 1, \ldots, n$ we have

$$g_i|_{v_1 v_2 \ldots v_n} = g_i|_{v_1 v_2 \ldots v_i}|_{v_{i+1} \ldots v_n} = 1.$$

We conclude that for every word containing the word $w = v_1 v_2 \ldots v_n$, i.e., for every word of the form $u_1 w u_2$, we have

$$g|_{u_1 w u_2} = g|_{u_1}|_w|_{u_2} = g_i|_w|_{u_2} = 1|_{u_2} = 1$$

for some i.

Thus, if an infinite word $x_1 x_2 \ldots \in \mathsf{X}^\omega$ contains $w = v_1 v_2 \ldots v_n$ as a subword, then $g|_{x_1 \ldots x_n} = 1$ for some n, i.e., $x_1 x_2 \ldots$ is g-rigid. But it is obvious that the set of infinite words containing a given finite word w has full measure. □

Let g be a homeomorphism of a compact topological space $\mathcal{X}$. A point $\xi \in \mathcal{X}$ is said to be *g-regular* if either $\xi^g \neq \xi$, or g fixes pointwise a neighborhood of ξ. If G is a homeomorphism group of $\mathcal{X}$, then a point $\xi \in \mathcal{X}$ is said to be G-regular, if it is g-regular for every $g \in G$.

Suppose that G is a countable homeomorphism group of a compact space $\mathcal{X}$. One can prove the following properties of G-regular points (see [GNS00] and [Nek04]):

1. The set of G-regular points is co-meager, i.e., is an intersection of a countable set of open dense sets.

2. Suppose that G is generated by a finite generating set S and that the action is minimal on $\mathcal{X}$, i.e., that every G-orbit is dense. Then for every G-regular point ξ the Schreier graph $\Gamma(G, G_\xi, S)$ is locally contained in the Schreier graph $\Gamma(G, G_\zeta, S)$ for every $\zeta \in \mathcal{X}$.

Here G_ζ denotes the stabilzier of ζ in G. A graph Γ_1 is *locally contained* in a graph Γ_2 if for every vertex v_1 of Γ_1 and every $R \in \mathbb{N}$ there exists a vertex v_2 of Γ_2 such that the ball in Γ_1 of radius R with center in v_1 is isomorphic as a labeled graph with the ball in Γ_2 of radius R with center in v_2. The balls are viewed as subgraphs of Γ_i with the induced graph structure.

Note that if $g|_{x_1 x_2 \ldots x_n}$ is trivial, then all the points of the cylindrical set $x_1 x_2 \ldots x_n \mathsf{X}_n^\omega$ are g-regular. Consequently, every g-rigid point of X^ω is g-regular. In particular, if G is a countable subgroup of $\mathcal{A}(\mathsf{X})$, then almost every point of X^ω is G-regular.

Theorem. *If a finitely-generated group $G \leq \mathcal{A}(\mathsf{X})$ is level-transitive, then for every point $\xi \in X^\omega$ the G-space G/G_ξ is amenable.*

Proof. Almost every G-orbit on X^ω consists of G-rigid sequences. Hence, for almost every $w \in \mathsf{X}^\omega$ and for every $g \in G$ the sequences w and w^g are *co-final*, i.e., are of the form $w = v_1 w'$ and $w^g = v_2 w'$, where $v_1, v_2 \in \mathsf{X}^n$ for some n and $w' \in \mathsf{X}_n^*$.

The co-finality equivalence relation is *hyperfinite*, i.e., is a union of an increasing sequence of measurable equivalence relations with finite equivalence classes. Namely, the co-finality relation is equal to $\bigcup_{n\geq 1} E_n$, where E_n is the equivalence relation consisting of the pairs $(w_1, w_2) \in \mathsf{X}^\omega \times \mathsf{X}^\omega$ such that $w_1 = v_1 w'$ and $w_2 = v_2 w'$ for some $w' \in \mathsf{X}_n^\omega$ and $v_1, v_2 \in \mathsf{X}^n$.

We see, therefore, that the G-orbit equivalence relation (the equivalence relation whose equivalence classes are the G-orbits) is a sub-relation of the hyperfinite co-finality relation, up to sets of measure zero. The group G acts by measure-preserving transformations on X^ω, hence, by Theorem 1 of [Kai97] the Schreier graphs $\Gamma(G, G_w, S)$ are amenable for almost every $w \in \mathsf{X}^\omega$. Moreover, since almost every point of X^ω is G-regular and the Schreier graphs $\Gamma(G, G_w, S)$, for G-regular w, are locally contained in every graph $\Gamma(G, G_\xi, S)$, the Schreier graph $\Gamma(G, G_\xi, S)$ is amenable for all $\xi \in \mathsf{X}^\omega$. □

4 Second construction

Let us show how the last theorem can be used to construct amenable actions of non-amenable groups. This construction was inspired by a "tree-wreathing" construction of S. Sidki.

Let G be any finitely-generated residually finite non-amenable group. It is known, that it acts faithfully on a spherically homogeneous rooted tree X^*. Take an additional letter $\$ \notin X_i$ and consider a new sequence $\mathsf{Y} = (X_1 \cup \{\$\}, X_2 \cup \{\$\}, \ldots)$. The tree X^* is in a natural way a sub-tree of the tree Y^*. We can extend the action of G on X^* to an action on Y^* in the following way. Suppose that $w \in \mathsf{Y}^*$ is arbitrary. If $w \in \mathsf{Y}^*$ does not belong to X^*, then it can be uniquely written in the form $w = v\$u$ for $v \in \mathsf{X}^*$ and $u \in \mathsf{Y}^*_{|v|+1}$. Then we set

$$w^g = v^g \$ u$$

for all $g \in G$. It is easy to see that we get in this way an action of G on Y^*, which extends the original action on X^*, and thus is also faithful. Note also that (in the case of a constant sequence X) the obtained action is finite-state if and only if the original action on X^* is finite-state.

The obtained action of G on Y^* is an action by almost finitary automorphisms, since every sequence $y_1 y_2 \ldots \in \mathsf{Y}^\omega$ containing the letter \$ is G-rigid. Hence, we get an embedding of G into $\mathcal{A}(\mathsf{Y})$. However, the action of G is not level-transitive on Y^*.

But it is easy to embed $G < \mathcal{A}(\mathsf{Y})$ into a level-transitive subgroup of $\mathcal{A}(\mathsf{Y})$. It is sufficient to take, for instance any level-transitive finitely-generated subgroup $H < \mathcal{A}(\mathsf{Y})$ and consider $F = \langle G, H \rangle$. Then F is a non-amenable subgroup of $\mathcal{A}(\mathsf{Y})$, and thus by the proved theorem, the F-space F/F_w is amenable for every w.

Probably, the simplest example of the group H is the infinite cyclic group generated by the *adding machine*. We identify the alphabets Y_i with the sets $\{0, 1, \ldots d_i - 1\}$, where $(d_1, d_2, \ldots)$ is the spherical index of Y^*, and define the *adding machines* a_n acting on Y_n^* by the recurrent rule

$$(iw)^{a_n} = \begin{cases} (i+1)w & \text{for } i = 0, 1, \ldots, d_{n+1} - 2 \\ 0w^{a_{n+1}} & \text{for } i = d_{n+1} - 1. \end{cases}$$

Then a_0 is an automorphism of Y^* generating a level-transitive cyclic group.

An explicit construction can be done in the following way. Consider the constant sequence $\mathsf{X} = (X, X, \ldots)$, where $X = \{0, 1\}$, and define three automorphisms a, b, c of X^* by the inductive rules

$$\begin{aligned} (0w)^a &= 1\left(w^b\right) & \qquad (1w)^a &= 0\left(w^b\right) \\ (0w)^b &= 0\left(w^a\right) & (1w)^b &= 1\left(w^c\right) \\ (0w)^c &= 0\left(w^c\right) & (1w)^c &= 1\left(w^a\right). \end{aligned}$$

It is known that the group generated by the automorphisms a, b, c is isomorphic to the free product $C_2 * C_2 * C_2$ of three groups of order 2, and thus is non-amenable. The definition of the transformations a, b and c implies that they are finite-state. This example of a three-state automaton was found by our students E. Muntyan and D. Savchuk. A proof, that it generates the free product $C_2 * C_2 * C_2$ can be found in [Nek05].

The above construction gives the following non-amenable group with amenable Schreier graphs. It is the group G generated by four transformations a, b, c, d acting on the tree defined over the alphabet $Y = \{0, 1, 2\}$ and satisfying the recursions

$$\begin{aligned} (0w)^a &= 1\left(w^b\right) & \qquad (1w)^a &= 0\left(w^b\right) & \qquad (2w)^a &= 2w \\ (0w)^b &= 0\left(w^a\right) & (1w)^b &= 1\left(w^c\right) & (2w)^b &= 2w \\ (0w)^c &= 0\left(w^c\right) & (1w)^c &= 1\left(w^a\right) & (2w)^c &= 2w \\ (0w)^d &= 1w & (1w)^d &= 2w & (2w)^d &= 0\left(w^d\right). \end{aligned}$$

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