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AMENABILITY AND PARADOXICAL DECOMPOSITIONS FOR PSEUDOGROUPS AND FOR DISCRETE METRIC SPACES

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Abstract. This is an expostion of various aspects of amenability and paradoxical decompositions for groups, group actions and metric spaces. First, we review the formalism of pseudogroups, which is well adapted to stating the alternative of Tarski, according to which a pseudogroup without invariant mean gives rise to paradoxical decompositions, and to defining a Følner condition. Using a Hall-Rado Theorem on matchings in graphs, we show then for pseudogroups that existence of an invariant mean is equivalent to the Følner condition; in the case of the pseudogroup of bounded perturbations of the identity on a discrete metric space, these conditions are moreover equivalent to the negation of the Gromov's so-called doubling condition, to isoperimetric conditions, to Kesten's spectral condition for related simple random walks, and to various other conditions. We define also the minimal Tarski number of paradoxical decompositions associated to a non-amenable group action (an integer ≥ 4), and we indicate numerical estimates (Sections II.4 and IV.2). The final chapter explores for metric spaces the notion of superamenability, due for groups to Rosenblatt.

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I. Introduction

The present exposition shows various aspects of amenability and non-amenability. Our initial motivation comes from a note on the *Banach-Tarski paradox* where Deuber, Simonovitz and Sós indicate one kind of paradoxical decomposition for metric spaces, in relation with what they call an "exponential growth" property [DeSS]. Our first purpose is to revisit their work which, in our view, relates paradoxical decompositions with *amenability* rather than with growth (see in particular Observation 33 below).

For this, we recall in Chapter II the formalism of set-theoretical pseudogroups which is well adapted to showing the many aspects of amenability: existence of invariant finitely

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additive measures, absence of paradoxical decomposition, existence of *Følner sets* and *isoperimetric estimates.* We also state one version of the basic *Tarski alternative :* a pseudogroup is either amenable or paradoxical.

In Chapter III, we specialize the discussion to metric spaces and pseudogroups of bounded perturbations of the identity; metric spaces, there, are discrete (except at the very end of the chapter). On one hand, this is an interesting class, with many examples given by finitely generated groups. On the other hand, it provides a convenient setting for proving Følner characterization as stated in Chapter II. We discuss also the *Kesten characterization in terms of simple random walks.*

For a group G which is not amenable, we estimate in Chapter IV the *Tarski number* $\mathcal{T}(G) \in \{4, 5, \dots, \infty\}$, an integer which indicates the minimal number of pieces involved in a paradoxical decomposition of G. It is known that $\mathcal{T}(G) = 4$ if and only if G has a subgroup which is free non abelian. We show that one has $5 \le \mathcal{T}(G) \le 34$ [respectively $6 \le \mathcal{T}(G) \le 34$] for some torsion-free groups [resp. for some torsion groups] constructed by Ol'shanskii [Ol1], and $6 \le \mathcal{T}\big(B(m,n)\big) \le 14$ for $B(m,n)$ a *Burnside group* on $m \ge 2$ generators of odd exponent $n \ge 665$ [Ady].

Building upon the seminal 1929 paper by von Neumann [NeuJ], Rosenblatt has defined for groups a notion of *superamenability.* He has shown that superamenable groups include those of subexponential growth, and it is not known whether there are others. In Chapter V, we investigate superamenability for pseudogroups and for discrete metric spaces; in particular, we describe a simple example of a graph which is both superamenable and of superexponential growth.

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II. Amenable pseudogroups

II.1. Pseudogroups

1. Definition. In the present set-theoretical context, a *pseudogroup* $\mathcal{G}$ of transformations of a set X is a set of bijections $\gamma : S \to T$ between subsets S, T of X which satisfies the following conditions (as listed, e.g., in [HS1]):

(i) the identity $X \to X$ is in $\mathcal{G}$,
(ii) if $\gamma : S \to T$ is in $\mathcal{G}$, so is the inverse $\gamma^{-1} : T \to S$,
(iii) if $\gamma : S \to T$ and $\delta : T \to U$ are in $\mathcal{G}$, so is their composition $\delta\gamma : S \to U$,
(iv) if $\gamma : S \to T$ is in $\mathcal{G}$ and if S' is a subset of S, the restriction $\gamma|S' : S' \to \gamma(S')$ is in $\mathcal{G}$,
(v) if $\gamma : S \to T$ is a bijection between two subsets S, T of X and
if there exists a *finite* partition $S = \sqcup_{1 \le j \le n} S_j$ with $\gamma|S_j$ in $\mathcal{G}$ for $j \in \{1, \dots, n\}$,
then γ is in $\mathcal{G}$ (where $\sqcup$ denotes a disjoint union).

Property (v) expresses the fact that $\mathcal{G}$ is closed with respect to *finite gluing up;* together with (iv), they express the fact that, for a bijection γ, being in $\mathcal{G}$ is in some sense a *local* condition.

For $\gamma : S \to T$ in $\mathcal{G}$, we write also $\alpha(\gamma)$ for the domain S of γ and $\omega(\gamma)$ for its range T. For "a pseudogroup $\mathcal{G}$ of transformations of a set X", we write shortly "a pseudogroup $(\mathcal{G}, X)$", or even "a pseudogroup $\mathcal{G}$".

2. Examples. (i) Any *action of a group* G on a set X generates a pseudogroup $\mathcal{G}_{G,X}$. More precisely, a bijection $\gamma : S \to T$ is in $\mathcal{G}_{G,X}$ if there exists a finite partition $S = \sqcup_{1 \le j \le n} S_j$ and elements $g_1, \dots, g_n \in G$ such that $\gamma(x) = g_j(x)$ for all $x \in S_j, j \in \{1, \dots, n\}$. If there exists such a γ, the subsets S, T of X are sometimes said to be G-equidecomposable (or "endlich zerlegungsgleich" in [NeuJ]).

In case $G = X$ acts on itself by left multiplications, we write $\mathcal{G}_G$ instead of $\mathcal{G}_{G,G}$.

(ii) *Piecewise isometries* of a metric space X constitute a pseudogroup $\mathcal{P}i\mathcal{I}s(X)$, generated (in the obvious way) by the partial isometries between subsets of X. Observe that it may be much larger than the pseudogroup associated as in the previous example with the group of isometries of X; see for example the metric space obtained from the real line by gluing two hairs of different length at two distinct points of the line.

(iii) For a metric space X, the pseudogroup $\mathcal{W}(X)$ of *bounded perturbations of the identity* consists of bijections $\gamma : S \to T$ such that $\sup_{x \in S} d(\gamma(x), x) < \infty$. This is the main example in [DeSS], where it is called the group of *wobbling bijections;* the notion seems to come from the important work by Laczkovich [Lacz]. See also Item $0.5.C_1''$ in [Gro3].

(iv) Given a pseudogroup $\mathcal{G}$ of transformations of a set X and a *subset* A of X, the set of bijections $\gamma \in \mathcal{G}$ with $\alpha(\gamma) \subset A$ and $\omega(\gamma) \subset A$ constitute a pseudogroup of transformations of A, denoted below by $\mathcal{G}_{(A)}$.

(v) From a pseudogroup $(\mathcal{G}, X)$ and an integer $k \ge 1$, one obtains a pseudogroup $\mathcal{G}_k$ of transformations of the direct product X_k of X and $\{1, \dots, k\}$, generated by the bijections of the form

$$\begin{cases} S \times \{j\} & \longrightarrow & T \times \{j'\} \\ (x, j) & \longmapsto & (\gamma(x), j') \end{cases}$$

where $\gamma : S \to T$ is in $\mathcal{G}$ and $1 \le j, j' \le k$.

3. Remarks. The above notion of pseudogroup of transformations is strongly motivated by the study of Banach-Tarski paradoxes, as shown by the first three observations below.

(i) The very definition of a paradoxical decomposition with respect to a group action involves the associated pseudogroup as in Example 2.i.

(ii) Pseudogroups are easily restricted on subsets as in Example 2.iv. This is important for the study of superamenability (see Chapter V below).

(iii) Pseudogroups are easily induced on oversets, as in Example 2.v. This is useful in the setting of a pseudogroup constituted by bijections with domains and range required to be in a given algebra (or σ-algebra) of subsets of X (for example the measurable sets of a measure space), and in corresponding variations on the Tarski alternative [HS1].

(iv) For a pseudogroup $(\mathcal{G}, X)$, the set

$$\mathcal{R} = \left\{ (x,y) \in X \times X \,\middle|\, \text{there exists } \gamma \in \mathcal{G} \text{ such that } x \in \alpha(\gamma) \text{ and } y = \gamma(x) \right\}$$

is an *equivalence relation.* A natural problem is to study the existence of measures μ on X such that, for each measurable subset A of X of measure zero, the saturated set $\{ x \in A \mid \text{there exists } a \in A \text{ with } (x,a) \in \mathcal{R} \}$ has also measure zero, see [CoFW], [Kai2], [Kai3].

(v) In a topological context, Conditions (iv) and (v) in Definition 1 are usually replaced by a condition involving restrictions to *open subsets*; see [Sac] and page 1 of [KoNo].

(vi) Consider a metric space X, the pseudogroup $\mathcal{W}(X)$ of Example 2.iii, and a subspace A of X. It is then remarkable (though straightforward to check) that the pseudogroup $\mathcal{W}(A)$ coincides with the restriction of $\mathcal{W}(X)$ to A in the sense of Example 2.iv.

II.2. Amenability and paradoxical decompositions - the Tarski alternative

Let $(\mathcal{G}, X)$ be a pseudogroup. We denote by $\mathcal{P}(X)$ the set of all subsets of X.

4. Definitions. A *$\mathcal{G}$-invariant mean* on X is a mapping $\mu : \mathcal{P}(X) \to [0,1]$ which is

(fa) finitely additive: $\mu(S_1 \cup S_2) = \mu(S_1) + \mu(S_2)$ for $S_1, S_2 \subset X$ with $S_1 \cap S_2 = \emptyset$,
(in) invariant: $\mu\big(\omega(\gamma)\big) = \mu\big(\alpha(\gamma)\big)$ for all $\gamma \in \mathcal{G}$,
(no) normalised: $\mu(X) = 1$.

More generally, for $A \subset X$, a *$\mathcal{G}$-invariant mean on X normalised on A* is a mapping $\mu : \mathcal{P}(X) \to [0, \infty]$ which satisfies Conditions (fa) and (in) above, as well as

(no') $\mu(A) = 1$.

The pseudogroup $\mathcal{G}$ is *amenable* if there exists a $\mathcal{G}$-invariant mean on X, and the triple $(\mathcal{G}, X, A)$ is *amenable* if there exists a $\mathcal{G}$-invariant mean on X normalised on A. These notions are essentially due to von Neumann [NeuJ].

5. Definition. A *paradoxical $\mathcal{G}$-decomposition of X* is a partition $X = X_1 \sqcup X_2$ such that there exist $\gamma_j \in \mathcal{G}$ with $\alpha(\gamma_j) = X_j$ and $\omega(\gamma_j) = X$ $(j = 1, 2)$.

A pseudogroup $(\mathcal{G}, X)$ is *paradoxical* if it has a paradoxical $\mathcal{G}$-decomosition, or equivalently (because of Theorem 7 below) if it is not amenable.

6. Remarks. (i) *There cannot exist such paradoxical $\mathcal{G}$-decomposition if $\mathcal{G}$ is amenable.*

This is obvious, because (with the notations of Definitions 4 and 5) one cannot have $1 = \mu(X) = \mu(X_1) + \mu(X_2) = 2$!

It is remarkable that there is no further obstruction, as Theorem 7 shows.

(ii) Let $\mathcal{G}, \mathcal{H}$ be two pseudogroups of transformations of the same set X, with $\mathcal{G} \subset \mathcal{H}$. If $\mathcal{H}$ is amenable, then so is $\mathcal{G}$; if $\mathcal{G}$ is paradoxical, then so is $\mathcal{H}$. This will be used for example in the proof of Theorem 25 (Item 36).

(iii) In short-hand, Definition 5 reads $2[X] \overset{!!}{=} [X]$. It has variations in the literature; for example, one may ask $(n+1)[X] \overset{!!}{\leq} n[X]$, or more precisely :

there exists an integer $n \geq 1$ and elements $\gamma_1, \ldots, \gamma_N \in \mathcal{G}$ such that

$|\{j \in \{1, \ldots, N\} \mid x \in \alpha(\gamma_j)\}| \geq n+1$ for all $x \in X$, namely $\sum_{j=1}^{k} [\alpha(\gamma_j)] \geq (n+1)[X]$, and

$|\{j \in \{1, \ldots, N\} \mid x \in \omega(\gamma_j)\}| \leq n$ for all $x \in X$, namely $\sum_{j=1}^{k} [\omega(\gamma_j)] \leq n[X]$.

Then Remark (i) still holds for the same obvious kind of reason. Indeed, the variation is equivalent to Definition 5, as can be seen either with manipulations à la Cantor-Bernstein (see for example [HS1]) or as a consequence of the following theorem.

7. Theorem (Tarski alternative). *Let $\mathcal{G}$ be a pseudogroup of transformations of a set X. Exactly one of the following holds :*

- *either $\mathcal{G}$ is amenable,*
- *or there exists a paradoxical $\mathcal{G}$-decomposition of X.*

Let moreover A be a non-empty subset of X and let $\mathcal{G}_{(A)}$ be the pseudogroup obtained by restriction of $\mathcal{G}$, as in Example 2.iv. Exactly one of the following holds :

- *either there exists a $\mathcal{G}$-invariant mean on X normalised on A,*
- *or there exists a paradoxical $\mathcal{G}_{(A)}$-decomposition of A.*

The theorem originates in Tarski's work : see [Tar3], as well as earlier papers by Tarski ([Tar1], [Tar2]).

One proof for pseudogroups has been written up in [HS1]. Its starting point is an application of the Hahn-Banach theorem, to the Banach space $\ell^\infty(X)$ of bounded real-valued functions on X, to the subspace $d^\infty(X)$ of finite linear combinations of functions of the form $\chi\big(\omega(\gamma)\big) - \chi\big(\alpha(\gamma)\big)$ for some $\gamma \in \mathcal{G}$ (where $\chi(A)$ denotes the characteristic function of A), and to the open cone $\mathcal{C}$ of functions $F \in \ell^\infty(X)$ such that $\inf_{x \in X} F(x) > 0$; one has to observe that $\mathcal{G}$ has an invariant mean if and only if $d^\infty(X) \cap \mathcal{C} = \emptyset$. This proof uses also ideas of Banach, Cantor-Bernstein, Hausdorff, König, Kuratowski and von Neumann.

We give here another proof, based on what we call the Hall-Rado theorem (Theorem 35), which is essentially the "König theorem" of [Wag]. More precisely, the first statement of Theorem 7 is a straightforward consequence of Theorems 25 and 32, and the second statement follows (see the sketch below).

Much more complete information on all this can be found in Wagon's book (see [Wag], in particular Corollary 9.2 on page 128). Important more recent work in this area include [DouF].

Let us sketch the proof of the second statement of the theorem. Assume that the pseudogroup $\mathcal{G}_{(A)}$ is not paradoxical, so that, by the first statement, there exists a $\mathcal{G}_{(A)}$-invariant mean $\mu_A : \mathcal{P}(A) \to [0,1]$. Define then a mapping $\mu : \mathcal{P}(X) \to [0, \infty]$ as follows; for a subset Y of X, if there exists a partition $Y = \sqcup_{1 \leq j \leq n} Y_j$ and elements $\gamma_1 : Y_1 \to B_1, \ldots, \gamma_n : Y_n \to B_n$ in $\mathcal{G}$ with $B_1, \ldots, B_n \subset A$, then set $\mu(Y) = \sum_{j=1}^{n} \mu_A(B_j)$; otherwise, set $\mu(Y) = \infty$. Then one checks that μ is well defined and that it is a $\mathcal{G}$-invariant mean on X normalised on A.

8. Remark. A famous theorem of E. Hopf can be expressed very much like Tarski's alternative.

Let $T : X \longrightarrow X$ be an ergodic non-singular transformation of a finite probability space $(X, \mathcal{B}, m)$, with m non-atomic. Let $[[T]]$ denote the set of all 1-1 non-singular transformations $\phi : U \to V$ such that $\phi(x)$ belongs to the T-orbit of x for all $x \in U$ (with $U, V \in \mathcal{B}$); this $[[T]]$ is the *full groupoid of* T of Katznelson and Weiss [KaWe, page 324]. For two measurable subsets A, B of X, say that A *is dominated by* B, and write $A \prec B$, if there exists a measurable subset B' of B with $m(B \setminus B') > 0$ and a bijective transformation $\phi : A \longrightarrow B'$ in $[[T]]$.

Hopf alternative. *(i) In the situation above, exactly one of the following holds :*

- *there exists a T-invariant probability measure on $(X, \mathcal{B})$ equivalent to m,*
- *one has $X \prec X$.*

(ii) Also, exactly one of the following holds :

- *there exists a T-invariant infinite measure on $(X, \mathcal{B})$ equivalent to m,*
- *one has $X \prec X$, and there exists $A \in \mathcal{B}$ with $m(A) > 0$ such that A is not dominated by A.*

In other words, (i) says that there is a finite invariant measure in the measure class m if and only if X itself is *not* "Hopf-compressible", and (ii) that there is an infinite invariant measure in the measure class m if and only if X is Hopf-compressible and *some* measurable subset of X of positive measure is not Hopf-compressible [Weis].

If there exists a T-invariant probability measure [respectively infinite measure] on $(X, \mathcal{B})$ equivalent to m, then T is said to be of *type* II_1 [resp. of *Type* II_∞].

II.3. The case of groups

For any group G, we consider first the pseudogroup $\mathcal{G}_G$ which is associated with the action of G on itself on the left, as in Example 2.i.

Let now G be a group generated by a finite set S. Let $\ell_S : G \to \mathbb{N}$ denote the corresponding word length function; thus ℓ_S associates to $g \in G$ the smallest integer $n \geq 0$ for which there exist $s_1, \ldots, s_n \in S \cup S^{-1}$ with $g = s_1 \ldots s_n$. Let d_L and d_R denote respectively the left and right invariant metrics on G defined by

$$\begin{aligned} d_L(x, y) &= \ell_S\left(x^{-1}y\right) \\ d_R(x, y) &= \ell_S\left(xy^{-1}\right) \end{aligned}$$

for all $x, y \in G$.

Besides $\mathcal{G}_G$, we consider also the pseudogroup $\mathcal{P}i\mathcal{I}s(G)$ of piecewise isometries of the metric space (G, d_L), as in Example 2.ii, as well as the pseudogroup $\mathcal{W}(G)$ of bounded perturbations of the identity of the metric space (G, d_R), as in Example 2.iii. It is easy to check that the pseudogroup $\mathcal{W}(G)$ does not depend on the choice of S.

9. Observation. *With the notations above, one has $\mathcal{G}_G = \mathcal{W}(G)$ for any finitely generated group G.*

Proof. It is obvious that $\mathcal{G}_G \subset \mathcal{W}(G)$. Conversely, let $\gamma : U \to V$ be in $\mathcal{W}(G)$. Set

$$k = \sup_{x \in U} d(\gamma(x), x)$$
$$B = \{ g \in G \mid \ell_S(g) \leq k \}$$

and observe that B is a finite subset of G. For each $g \in B$, set

$$U_g = \{ x \in U \mid \gamma(x) = gx \}.$$

One has $U = \sqcup_{g \in B} U_g$ and $\gamma(x) = gx$ for all $x \in U_g$. Hence $\gamma \in \mathcal{G}_G$. □

It is clear that $\mathcal{G}_G \subset \mathcal{P}i\mathcal{I}s(G)$. It is also clear that $\mathcal{G}_G \neq \mathcal{P}i\mathcal{I}s(G)$ in general (example : for $G = \mathbb{Z}$ generated by $\{1\}$, the isometry $n \mapsto -n$ is not in $\mathcal{G}_{\mathbb{Z}}$).

10. Definition. A group G is *amenable* if the pseudogroup $\mathcal{G}_G$ is amenable.

If G is finitely generated, the previous observation shows that one may equivalently define G to be amenable if the pseudogroup $\mathcal{W}(G)$ is amenable.

11. On the class of amenable groups. Amenability may be viewed as a finiteness condition. One of the main problems is to understand various classes of amenable groups, for example those which are finitely generated or finitely presented. (Recall that a group is amenable if and only if all its finitely generated subgroups are amenable; see Theorem 1.2.7 in [Gre1] and Observation 19 below.)

The following question, implicit in [NeuJ], was formulated explicitely by Day, at the end of § 4 in [Day1] : does every non-amenable group contain a free group on 2 generators ? As much as we know and despite several misleading allusions in the literature to some "von Neumann conjecture", von Neumann himself has *never conjectured* that a non-amenable group should contain a non-abelian free subgroup !

Day's question was answered negatively by A. Yu. Ol'shanskii [Ol1], Adyan [Ady2] and Gromov [Gro2, Corollary 5.6.D]; the first two use cogrowth criteria (see Item 52 below) and Gromov uses Property (T). For infinite groups, this *Property (T) of Kazhdan* [Kaz] is (among other things) a strong form of non-amenability : see [Sch] and [CoWe]. However, when restricted to the class of *linear groups* (i.e. of groups which have faithful finite-dimensional linear representations), Day's question can be answered positively : this follows from an important result due to Tits [Tit].

M. Day has defined the class EG of "elementary amenable groups", which is the smallest class of groups which contains finite groups and abelian groups, and which is closed under the four operations of (*i*) taking subgroups, (*ii*) forming factor groups, (*iii*) group extensions and (*iv*) upwards directed unions. He has asked (again in [Day1]) whether the class EG coincides with the class AG of all amenable groups (see also [Cho]).

Today, we know that there are finitely generated groups in AG which are not in EG; this has first been shown using growth estimates ([Gri2], [Gri3]), and more recently by an elegant argument of Stepin (see [Ste], based on [Gri2]).

One knows also finitely *presented* groups in AG which are not in EG; more precisely, the finite presentation

$$G = \left\langle a, b, c, d, t \;\middle|\; \begin{array}{l} a^2 = b^2 = c^2 = d^2 = bcd = (ad)^4 = (adacac)^4 = 1 \\ t^{-1}at = aca \quad t^{-1}bt = d \quad t^{-1}ct = b \quad t^{-1}dt = c \end{array} \right\rangle$$

defines an amenable group which is not elementary amenable ([Gri6], [Gri7]).

12. Bartholdi's presentation. It has later been shown that the group G of [Gri6] has a presentation with two generators only (namely a and t) and four relations of total length $109 = 2 + 19 + 32 + 56$. Here are Bartholdi's computations, where T stands for t^{-1}.

The relations $c = aTata$, $d = tcT$ and $b = Tct$ show first that the relations $c^2 = d^2 = b^2 = 1$ may be deleted in the presentation above, and second that the generators b, c, d may also be deleted. Thus

$$G = \left\langle a, t \;\middle|\; \begin{array}{c} a^2 = TctctcT = (atcT)^4 = (atcTacac)^4 = 1 \\ T^2ct^2 = tcT \end{array} \right\rangle$$

where c holds for $aTata$. The relation $TctctcT = 1$ implies $T^2ctctc = 1 = tcT^2ctc$ (by conjugation), hence also (using $c^{-1} = c$)

$$1 \;=\; \left(T^2ctctc\right)\left(tcT^2ctc\right)^{-1} \;=\; T^2ct^2\left(tcT\right)^{-1}$$

using free simplifications, so that the relation $T^2ct^2 = tcT$ may also be deleted. Finally, one observes that $atcT$ is conjugate to $Tatc = (Tata)^2$ so that $(atcT)^4 = 1$ may be written $(Tata)^8 = 1$, and one observes also that $atcTacac$ is equal to $ataTataTaaTataaaTata$, so is conjugate to $T^2ataTat^2aTata$. One obtains finally Bartholdi's presentation

$$G = \left\langle a, t \;\middle|\; a^2 = TaTatataTatataTataT = (Tata)^8 = (T^2ataTat^2aTata)^4 = 1 \right\rangle.$$

13. Categorical considerations. For a given integer k, let F_k be the free group on k generators $\{s_1, \ldots, s_k\}$ and let X_k denote the space of all *marked groups on k generators,* namely of all data $F_k \twoheadrightarrow \Gamma$, where $\twoheadrightarrow$ indicates a homomorphism *onto.* There is an appropriate topology on X_k, for which two quotients $\pi : F_k \twoheadrightarrow \Gamma$ and $\pi' : F_k \twoheadrightarrow \Gamma'$ are "near" each other if the corresponding Cayley graphs have balls of "large" radius around the unit element which are isomorphic. This makes X_k a compact space; one shows for example that the closure of the subset of X_k corresponding to finite groups contains the subset of X_k corresponding to residually finite finitely presented groups. For details, see [Gri2], [Cha] and [Ste].

It would be interesting to find pairs (Y, Z) where

- Y is a compact subspace of X_k,
- Z is a "small" (e.g. countable) subset of Y, consisting of amenable groups,

- $Y \setminus Z$ consists of non-elementary amenable groups, or more generally the set of elementary amenable groups in $Y \setminus Z$ is of first category.

The point is that the space Y contains a dense G_δ consisting of amenable groups which are not elementary amenable. (As usual a G_δ in Y is a countable intersection of open subsets of Y.)

One such pair has been constructed in [Gri2] and analized in [Ste], with Z a countable set of virtually 2-step solvable groups and with $Y \setminus Z$ consisting of infinite torsion groups. Understanding other such pairs would probably help us understanding the closures of AG_k and of EG_k in X_k, where AG_k [respectively EG_k] denotes the subspace of X_k containing marked groups $\pi : F_k \twoheadrightarrow \Gamma$ with Γ amenable [resp. elementary amenable].

14. Variation on one question of Day. Let us denote by BG the smallest class of groups containing finitely generated groups of subexponential growth (see Definition 64) and closed with respect to the four operations of Day listed in 11 above, namely with respect to (i) taking subgroups, (ii) forming factor groups, (iii) group extensions and (iv) upwards directed unions.

Question: *does one have BG=AG ?*

15. Other definitions of amenability for groups; topological groups. The natural setting for amenability of groups is that of topological groups, mainly locally compact groups. A substancial part of the theory consists in showing the equivalence of a large number of definitions.

Let G be a *Hausdorff topological* group. Denote by $\mathcal{C}^b(G)$ the Banach space of bounded continuous functions from G to $\mathbb{C}$, with the supremum norm. For $\xi \in \mathcal{C}^b(G)$ and $g \in G$, let ${}_g\xi \in \mathcal{C}^b(G)$ be the function $x \to f(g^{-1}x)$. Denote by $\mathcal{UC}^b(G)$ the closed subspace of $\mathcal{C}^b(G)$ of functions ξ for which the mapping $g \mapsto {}_g\xi$ from G to $\mathcal{C}^b(G)$ is continuous. The following are known to be equivalent (see Theorem 3 in [Day2] and Theorem 4.2 in [Ric2]) :

- there exists a left-invariant mean on $\mathcal{UC}^b(G)$,
- any continuous action $G \times Q \to Q$ of G by affine transformations of a non-empty compact convex subset Q of a Hausdorff locally convex topological vector space has a fixed point.

The group G is *amenable* if these properties hold. In case G is assumed to be locally compact, here is a short list of other equivalent properties :

- there exists a left-invariant mean on $\mathcal{C}^b(G)$,
- there exists a left-invariant mean on $L^\infty(G)$,
- the unit representation of G is weakly contained in the left regular representation of G on $L^2(G)$,
- for any continuous action $G \times X \to X$ of G by homeomorphisms of a non-empty compact space X, there exists a G-invariant probability measure on X.

The last point, on G-invariant measures, goes back to a paper by Bogolyubov, see [Bogl], quoted by Anosov [Ano]. This paper, published in Ukrainian in 1939, has remained unnoticed; the paper does *not* quote von Neumann [NeuJ], and it is conceivable that Bogolyubov has introduced independently the notion of amenability. About relations between amenability, growth and existence of invariant measures, we would also like to quote [Bekl].

The list above is very far from being complete ! (See 16; other items could be : several formulations of the Følner property for locally compact groups, the Reiter-Glicksberg property, the existence of approximate units in the Fourier algebra, ...) See, e.g., the books [Gre1], [Pat] and [Wag], as well as [Rei, Chapter 8], [Eym2], [Zim, Chapter 4], [Wag, in particular Theorem 10.11] and [Lub, Chapter 2]. In case of a countable group (with the discrete topology), here is the most recent characterization of amenability with which one of the authors has been involved : a countable group G is amenable if and only if, for any action of G by homeomorphisms on the Cantor discontinuum K, there exists a probability measure on K which is invariant by G [GiH2].

We would like to point out that some attention has been given to topological groups which are not locally compact (in [Ric2, § 4] among other places). For example, let $\mathcal{U}(\mathcal{H})_{st}$ be the group of unitary operators on a separable infinite dimensional Hilbert space $\mathcal{H}$, with the *strong topology;* then $\mathcal{U}(\mathcal{H})_{st}$ is amenable, namely there exists a left invariant mean on $\mathcal{UC}^b\left(\mathcal{U}(\mathcal{H})_{st}\right)$, but there does not exist any left invariant mean on $\mathcal{C}^b\left(\mathcal{U}(\mathcal{H})_{st}\right)$. Moreover, this group does have closed subgroups which are not amenable; indeed, if $\mathcal{H} = \ell^2(F_n)$ for a free group F_n of rank $n \geq 2$, then $\mathcal{U}(\mathcal{H})_{st}$ has clearly a discrete subgroup isomorphic to F_n, as observed in [Har3]. Here is another example involving non locally compact topologies; let G be the group of real points of an $\mathbb{R}$-algebraic group and let Γ be a subgroup of G which is dense for the *Zariski topology;* if Γ is amenable, so is G (see [Moo], and Theorem 4.1.15 in [Zim]).

Let us mention the following : for a locally compact group G which is almost connected (this means that the quotient of G by the connected component of 1 is compact), the three properties

G is amenable,

G does not contain a discrete subgroup which is free on 2 generators,

$G/r(G)$ is compact,

are equivalent. This is due to Rickert : Theorem 5.5 in [Ric2], building on [Ric1]; see also Theorem 3.8 in [Pat]. Recall that the *solvable radical* $r(G)$ of a locally compact group G is the largest connected closed normal solvable subgroup of G [Iwa]. (One may define similarly the *amenable radical* of G as the largest amenable closed normal subgroup of G; see Lemma 1 of § 4 in [Day1] and Proposition 4.1.12 in [Zim].)

This result of Rickert reduces in some sense the problem of understanding the class of amenable locally compact groups to totally disconnected groups; we believe moreover that the most important (and difficult) part of the problem is that which concerns finitely generated groups.

16. Cohomological definitions of amenability. There are various (co)homological characterizations of amenability.

One is that of Johnson : a group G is amenable if and only if $H^1(\ell^1(G), M^*)$ is reduced to $\{0\}$ whenever M^* is a G-module dual to some Banach G-module M [Joh]. It follows that the bounded cohomology of an amenable group is always reduced to $\{0\}$; this is given by Gromov (Section 3.0 in [Gro1]) together with a reference to an unpublished explanation of Philip Trauber - hence the name “Trauber theorem”.

Another one is in terms of “uniformly finite homology”; it applies to finitely generated groups, and indeed to metric spaces in a much broader class. Such a space X is *not* amenable if and only if the group $H_0^{uf}(X)$ is reduced to $\{0\}$ (in this statement, one may take $\mathbb{R}$ as coefficients, or equivalently $\mathbb{Z}$); this is one way to express that Følner condition does *not* hold in X [BlW1].

It seems also appropriate to quote here a theorem of Brooks : let G be the covering group of a normal covering M of a compact manifold X; then G is amenable if and only if 0 is in the spectrum of the Laplace-Beltrami operator acting on the space of square-integrable functions on M (see [Bro], or the exposition in [Lot]).

There are other conditions in terms of other “coarse” (co)homology theories of the groups, or in terms of K-theory of appropriate algebras associated to the group (see various preprints by G. Elek, including [Ele2]).

Let us mention that there are interesting cohomological *consequences* of amenability. For example, let G be a group which has an Eilenberg-MacLane space $K(G,1)$ which is a finite complex; if G is amenable, then G has Euler characteristic $\chi(G) = 0$ (a particular case of Corollary 0.6 of Cheeger and Gromov [ChGr], who use ℓ^2-cohomology methods, and also a result of B. Eckmann, who uses other methods [Eck]). Also, let G be the fundamental group of some closed 4-manifold M; if G is infinite and amenable, then $\chi(M) \geq 0$ [Eck].

17. Variations on amenability of groups. There are standard variations on the pseudogroup $\mathcal{G}_G$ and the notion of amenability.

One is to consider the pseudogroup $\mathcal{G}_{G\times G}$ associated as in Example 2.i with the action of $G \times G$ on G defined by $(x, y) \circ g = xgy^{-1}$. It is classical that $\mathcal{G}_{G\times G}$ is amenable if and only if $\mathcal{G}_G$ is amenable. In other words : G has a left invariant mean if and only if G has a two-sided invariant mean (Lemmas 1.1.1 and 1.1.3 in [Gre1]).

Another variation is to consider the action of G on $G \setminus \{1\}$ defined by $x \circ g = xgx^{-1}$ and the notion of *inner amenability* for a group. It is obvious that an amenable group is inner amenable. Straightforward examples (such as non-trivial direct products of free groups and amenable groups) show that there are non-amenable groups which are inner amenable. More on this in [BeHa], [Eff], [GiH1] and [HS2].

A third variation is to consider a subgroup H of G and the pseudogroup $\mathcal{G}_{G/H}$ associated with the natural action of G on G/H. The subgroup H is said to be *co-amenable* in G if $\mathcal{G}_{G/H}$ is amenable. There is a comprehensive analysis of this notion in [Eym1]; see also [Bekk], in particular Theorem 2.3. In case $G = F_m$ is a free group of finite rank, a criterion for co-amenability of a subgroup in terms of *cogrowth* is given in [Gri1] (see Item 52 below). One may generalize actions of G on G/H to actions of G on locally compact spaces; co-amenability of H is then a particular case of a notion of amenability for actions known as *amenability in the sense of Greenleaf* [Gre2].

The notion of amenability for a group and that of co-amenability for a subgroup may both be viewed as particular cases of a notion for G-mappings, for which we refer to [AnaR]. In case of a group G with the discrete topology, it can be defined as follows. Let X, Y be two Borel spaces given with measure classes μ, ν and with actions of G by non-singular invertible Borel mappings, and let $\phi : X \to Y$ be a surjective Borel mapping such that

$\phi_*(\mu) = \nu$; thus there is a canonical linear isometric mapping by which we identify the Banach space $L^\infty(Y, \nu)$ to a closed G-invariant subspace of $L^\infty(X, \mu)$. Say the mapping ϕ is *amenable* if there exists a G-equivariant linear mapping $E : L^\infty(X, \mu) \to L^\infty(Y, \nu)$ which is a conditional expectation, namely which is positive and which restricts to the indentity on $L^\infty(Y, \nu)$. Example 1 : $X = G$ and Y is reduced to one point; then $X \to Y$ is amenable if and only if G is amenable. Example 2 : $X = G/H$ for a subgroup H of G and Y is reduced to a point; then $X \to Y$ is amenable if and only if H is co-amenable in G. Example 3 : $X = G \times Z$ for a G-space Z (with G acting from the left on itself and diagonally on the product $G \times Z$); then the projection $G \times Z \to Z$ is amenable if and only if the action of G on Z is amenable *in the sense of Zimmer* [Zim, Section 4.3].

There are other notions, including the three following ones : *K-amenability* [Cun], *weak amenability* à la Cowling-Haagerup [CowH], and *a-T-menability* à la Gromov. (See 7.A and 7.E in [Gro3], and [BekCV]; in fact Gromov rediscovered the class of groups having “Property 3B” of Akemann and Walter in [AkWa].)

II.4. Tarski number of paradoxical group actions

Consider more generally the pseudogroup $\mathcal{G}_{G,X}$ associated with a group action $G \times X \to X$ (see again Example 2.i).

18. Definition. For $\gamma : S \to T$ in $\mathcal{G}_{G,X}$, define the Tarski number of γ as the smallest “number of pieces” n such that there exists a partition $S = \sqcup_{1 \le j \le n} S_j$ and elements $g_1, \ldots, g_n$ in G with $\gamma(x) = g_j(x)$ for all $x \in S_j, j \in \{1, \ldots, n\}$.

The *Tarski number* of a paradoxical $\mathcal{G}_{G,X}$-decomposition

$$X = X_1 \bigsqcup X_2 \quad , \quad \gamma_1 : X_1 \to X \quad , \quad \gamma_2 : X_2 \to X$$

as above is the sum of the Tarski number of γ_1 and of that of γ_2. It is clear that such a sum is an integer ≥ 4.

When $\mathcal{G}_{G,X}$ is not amenable, we define the *Tarski number* $\mathcal{T}(G, X)$ of the action $G \times X \to X$ as the minimum of the Tarski numbers of the paradoxical $\mathcal{G}_{G,X}$-decompositions of X; when $\mathcal{G}_{G,X}$ is amenable, we set $\mathcal{T}(G, X) = \infty$. For a group G acting on itself by left multiplication, we write $\mathcal{T}(G)$ rather than $\mathcal{T}(G, G)$.

19. Observation. Let G be a group given together with a subgroup G' and a quotient group G''. It is straightforward that one has

$$\mathcal{T}(G) \leq \mathcal{T}(G')$$
$$\mathcal{T}(G) \leq \mathcal{T}(G'').$$

For example, for the first of these inequalities, view G as a disjoint union of cosets of G'.

Each group G has a finitely generated subgroup G' such that $\mathcal{T}(G') = \mathcal{T}(G)$. Indeed, assuming G to be non-amenable, consider a paradoxical decomposition

$$G = X_1 \sqcup \ldots \sqcup X_m \sqcup Y_1 \ldots \sqcup Y_n = g_1 X_1 \sqcup \ldots \sqcup g_m X_m = h_1 Y_1 \sqcup \ldots \sqcup h_n Y_n$$

containing $m + n = \mathcal{T}(G)$ pieces (where $X_1, \ldots, X_m, Y_1, \ldots, Y_n$ are subsets of G and $g_1, \ldots, g_m, h_1, \ldots, h_n$ are elements of G). Let G' be the subgroup of G generated by $\{g_1, \ldots, g_m, h_1, \ldots, h_n\}$. Set $X'_i = X_i \cap G'$ for all $i \in \{1, \ldots, m\}$ and $Y'_j = Y_j \cap G'$ for all $j \in \{1, \ldots, n\}$. Then

$$G' = X'_1 \sqcup \ldots \sqcup X'_m \sqcup Y'_1 \ldots \sqcup Y'_n = g_1 X'_1 \sqcup \ldots \sqcup g_m X'_m = h_1 Y'_1 \sqcup \ldots \sqcup h_n Y'_n$$

so that $\mathcal{T}(G') \leq \mathcal{T}(G)$. With the first inequality of the present observation, this shows that $\mathcal{T}(G') = \mathcal{T}(G)$. (One may observe a fortiori that $X'_1, \ldots, Y'_n$ are non-empty.) It follows that one has

$$\mathcal{T}(G) = \inf \big(\mathcal{T}(G')\big)$$

where the infimum is taken over all finitely generated subgroups G' of G.

It should be interesting to study how the Tarski number behaves with respect to other group theoretical constructions such as extensions and HNN-constructions. In particular, for the latter, we have in mind some presentations of the Richard Thompson's F group [CaFP]; recall that F is a group which does not have non-abelian free subgroups, which is a HNN-extension of itself [BrGe], that F is inner-amenable [Jol], that F has non-abelian free subsemigroups so that it is not supermanenable (see Chapter V below), and that one does not know whether F is amenable or not.

20. Proposition (Jonsson, Dekker). *For a group G, one has $\mathcal{T}(G) = 4$ if and only if G contains a non-abelian free subgroup.*

Proof. For the free group F_2 on 2 generators g and h, it is classical that $\mathcal{T}(F_2) = 4$; see, e.g., Figure 4.1 in [Wag]. We recall this as follows. Set

$$\begin{aligned} A_1 &= W(g) \\ A_2 &= W(g^{-1}) \\ B_1 &= W(h) \cup \{1, h^{-1}, h^{-2}, \ldots\} \\ B_2 &= W(h^{-1}) \setminus \{h^{-1}, h^{-2}, \ldots\} \end{aligned}$$

where $W(x)$ denotes the subset of F_2 consisting of reduced words on $\{g, h\}$ with x as first letter on the left, for $x \in \{g, g^{-1}, h, h^{-1}\}$. Then

$$F_2 = A_1 \bigsqcup A_2 \bigsqcup B_1 \bigsqcup B_2 = A_1 \bigsqcup g A_2 = B_1 \bigsqcup h B_2.$$

It follows that $\mathcal{T}(F_2) = 4$.

Observation 19 shows that $\mathcal{T}(G) = 4$ for any group G containing a subgroup isomorphic to F_2.

Conversely, let G be a group with $\mathcal{T}(G) = 4$, so that there exist subsets X_1, X_2, Y_1, Y_2 and elements g_1, g_2, h_1, h_2 in G such that

$$G = X_1 \bigsqcup X_2 \bigsqcup Y_1 \bigsqcup Y_2 = g_1 X_1 \bigsqcup g_2 X_2 = h_1 Y_1 \bigsqcup h_2 Y_2.$$

Set $g = g_1^{-1}g_2$ and $h = h_1^{-1}h_2$. Then, one has successively

$$\begin{aligned} X_1 \;&=\; G \setminus gX_2 \;=\; gX_1 \bigsqcup gY_1 \bigsqcup gY_2 \\ X_1 \;&\supset\; gX_1 \;\supset\; \dots \;\supset\; g^{k-1}X_1 \;\supset g^kY_j \qquad (k \geq 1 \quad \text{and} \quad j = 1,2) \\ X_2 \;&=\; G \setminus g^{-1}X_1 \;=\; g^{-1}X_2 \bigsqcup g^{-1}Y_1 \bigsqcup g^{-1}Y_2 \\ X_2 \;&\supset\; g^{-1}X_2 \;\supset\; \dots \;\supset g^{-k+1}X_2 \;\supset\; g^{-k}Y_j \qquad (k \geq 1 \quad \text{and} \quad j = 1,2) \end{aligned}$$

so that

$$g^kY_j \;\subset X_1 \cup X_2 \qquad \text{for all} \qquad k \in \mathbb{Z}\,,\, k \neq 0 \quad \text{and} \quad j = 1,2.$$

One has similarly

$$h^kX_j \;\subset Y_1 \cup Y_2 \qquad \text{for all} \qquad k \in \mathbb{Z}\,,\, k \neq 0 \quad \text{and} \quad j = 1,2.$$

Hence g and h generate in G a free subgroup of rank 2, by a classical lemma going back essentially to F. Klein, and sometimes known as the "table-tennis lemma" (see, e.g., [Har4]).

The argument above is our rephrasing of the proof of Theorem 4.8 in [Wag]. □

Proposition 20 is an unpublished work from the 40's by B. Jonsson (a student of Tarski) and is a particular case of results of Dekker published in the 50's. For precise references, see the Notes of Chapter 4 in [Wag].

Let us also mention that, for a group G containing a non abelian free group and for an action $G \times X \to X$ with stabilizers $\{g \in G \,|\, gx = x\}$ which are abelian for all $x \in X$, the corresponding Tarski number is also given by $\mathcal{T}(G, X) = 4$ (Theorem 4.5 in [Wag]).

21. Proposition. *For a non-amenable torsion group G, one has $\mathcal{T}(G) \geq 6$.*

Proof. By Proposition 20 we know that $\mathcal{T}(G) \geq 5$. We assume that $\mathcal{T}(G) = 5$, and we will reach a contradiction.

The hypothesis implies that there exist subsets X_1, X_2, Y_1, Y_2, Y_3 and elements g_1, g_2, h_1, h_2, h_3 in G such that

$$G \;=\; X_1 \bigsqcup X_2 \bigsqcup Y_1 \bigsqcup Y_2 \bigsqcup Y_3 \;=\; g_1X_1 \bigsqcup g_2X_2 \;=\; h_1Y_1 \bigsqcup h_2Y_2 \bigsqcup h_3Y_3.$$

Let n denote the order of $g \doteq g_1^{-1}g_2$. As in the proof of Proposition 11, one has

$$X_1 \;\supset\; gX_1 \;\supset\; \dots \;\supset\; g^{n-1}X_1 \;\supset\; g^n \left(Y_1 \bigsqcup Y_2 \bigsqcup Y_3 \right).$$

But now $g^n = 1$ and this is absurd. Hence $\mathcal{T}(G) > 5$. □

22. Question. Does there exist a group G with Tarski number $\mathcal{T}(G)$ equal to 5 ? to 6 ? More generally, what are the possible values of $\mathcal{T}(G)$?

II.5. Følner condition for pseudogroups

Let $(\mathcal{G}, X)$ be a pseudogroup of transformations. For a subset $\mathcal{R}$ of $\mathcal{G}$ and a subset A of X, we define the *$\mathcal{R}$-boundary of A* as

$$\partial_{\mathcal{R}} A = \left\{ x \in X \setminus A \;\middle|\; \begin{array}{c} \text{there exists } \rho \in \mathcal{R} \cup \mathcal{R}^{-1} \text{ such that} \\ x \in \alpha(\rho) \quad \text{and} \quad \rho(x) \in A \end{array} \right\}.$$

23. Definition. *The pseudogroup $(\mathcal{G}, X)$ satisfies the Følner condition if*

for any finite subset $\mathcal{R}$ of $\mathcal{G}$ and for any real number $\epsilon > 0$
there exists a finite non-empty subset $F = F(\mathcal{R}, \epsilon)$ of X
such that $|\partial_{\mathcal{R}} F| < \epsilon |F|$

where $|F|$ denotes the cardinality of the set F.

24. Ahlfors and Følner. Ideas underlying the Følner condition go back at least to Ahlfors. (Følner does *not* refer to this work.) Ahlfors defines an open Riemann surface S to be *regularly exhaustible* if, for some complete metric g in the conformal class defined by the complex structure of S, there exists a nested sequence $\Omega_1 \subset \Omega_2 \subset \dots$ of domains with smooth boundaries such that $\bigcup_{n \geq 1} \Omega_n$ is the whole surface and such that

$$\lim_{n \to \infty} \frac{|\partial \Omega_n|_g}{|\Omega_n|_g} = 0$$

where $|\Omega|_g$ denotes the area of a domain Ω and where $|\partial\Omega|_g$ denotes the length of its boundary, both with respect to g. (A lemma of Ahlfors shows that this does not depend on the choice of g.) These sequences may be used to define averaging processes, as Ahlfors did first and as Følner did later.

Using this notion, Ahlfors has developped a geometric approach to the Nevanlinna theory of distribution of values of meromorphic functions, known as Ahlfors theory of covering surfaces. In particular, he gave a generalization of the second main theorem of Nevanlinna on defect. (See Section 25 in Chapter III of [Ahl]; see also Chapter XIII in [Nev], Chapter 5 in [Hay], Theorem 6.5 on page 1223 of [Oss], [Sto] and [ZoKe].)

Amenability of coverings of Riemann surfaces can also be expressed in terms of Teichmüller spaces [McM2].

25. Theorem. *A pseudogroup of transformations is amenable if and only if it satisfes the Følner condition.*

Følner's original proof (for a group acting on itself by left multiplications) goes back to 1955 [Fol]. The proof has been simplified by Namioka [Nam] (who generalized Følner's result to one-sided cancellative semi-groups), and extended to group actions by Rosenblatt [Ros1]; the best place to read it is probably Section 2.1 of [Co1]. In case of a group G acting by conjugation on $G \setminus \{1\}$, the proof can also be found in [BeHa], and it applies *verbatim* to an action of G on any set X. All these references use essentially techniques of functional analysis. (See also Wagon's comment about the implication (6) $\Longrightarrow$ (1) in Theorem 10.11 of [Wag].)

The proof below, in Items 26 and 36, uses completely different techniques.

26. Beginning of the proof of Theorem 25. We prove here the implication "Følner condition $\Rightarrow$ existence of an invariant mean".

Let $\mathcal{M}(X)$ denote the set of all means on X, namely of all finitely additive probability measures on X (see Conditions (fa) and (no) in Definition 4). Let $\ell^\infty(X)$ denote the Banach space of all bounded functions on X, with the norm of uniform convergence; it is standard[1] that $\mathcal{M}(X)$ can be identified with a subset of the unit ball in the dual space of $\ell^\infty(X)$. It is also standard that the weak*-topology makes $\mathcal{M}(X)$ into a compact space.

For each finite non-empty subset F of X, we consider the mean

$$\mu_F : \begin{cases} \mathcal{P}(X) & \longrightarrow & [0,1] \\ A & \longmapsto & \dfrac{|A \cap F|}{|F|} \end{cases}$$

in $\mathcal{M}(X)$. Consider also the set

$$\mathcal{N} = \{ (\mathcal{R}, \epsilon) \in \mathcal{G} \times \mathbb{R} \mid \mathcal{R} \quad \text{is finite and} \quad \epsilon > 0 \}$$

ordered by

$$(\mathcal{R}, \epsilon) \le (\mathcal{R}', \epsilon') \qquad \text{if} \qquad \mathcal{R} \subset \mathcal{R}' \quad \text{and} \quad \epsilon \ge \epsilon'.$$

Notations being as in the definition of the Følner condition (which is now assumed to hold),

$$(*) \qquad \big(\mu_{F(\mathcal{R},\epsilon)}\big)_{(\mathcal{R},\epsilon)\in\mathcal{N}}$$

becomes a *net*. By compacity of $\mathcal{M}(X)$, this net has a cluster point, say μ (we use the terminology of [Kel, Chapter 2]). The proof consists in showing that μ is $\mathcal{G}$-invariant; in other words, given a subset A of X and a transformation γ in $\mathcal{G}$ with $A \subset \alpha(\gamma)$, one has to show that $\mu\big(\gamma(A)\big) = \mu(A)$.

We choose a number $\delta > 0$. As μ is a cluster point of the family $(*)$, there exists $(\mathcal{R}, \epsilon) \in \mathcal{N}$ such that

$$\begin{array}{ll} (i) & (\mathcal{R}, \epsilon) \ge (\{\gamma\}, \delta), \quad \textit{i.e.}, \quad \mathcal{R} \ni \gamma \quad \text{and} \quad \epsilon \le \delta, \\ (ii) & |\mu_{F(\mathcal{R},\epsilon)}(A) - \mu(A)| \le \delta, \\ (iii) & \big|\mu_{F(\mathcal{R},\epsilon)}\big(\gamma(A)\big) - \mu\big(\gamma(A)\big)\big| \le \delta. \end{array}$$

From now on, we write F instead of $F(\mathcal{R}, \epsilon)$. Define

$$\begin{aligned} A_{i,i} &= \{ a \in A \,|\, a \in F \quad \text{and} \quad \gamma(a) \in F \} \\ A_{i,o} &= \{ a \in A \,|\, a \in F \quad \text{and} \quad \gamma(a) \in \partial_{\mathcal{R}} F \} \\ A_{o,i} &= \{ a \in A \,|\, a \in \partial_{\mathcal{R}} F \quad \text{and} \quad \gamma(a) \in F \} \\ A_{o,o} &= \{ a \in A \,|\, a \notin F \quad \text{and} \quad \gamma(a) \notin F \} \end{aligned}$$

[1] See footnote 37 in [NeuJ], where von Neumann refers in turn to Lebesgue's "Leçons sur l'intégration" (1905).

(think of "inside" for "i" and of "outside" for "o"). Observe that $A = A_{i,i} \sqcup A_{i,o} \sqcup A_{o,i} \sqcup A_{o,o}$, with the first three sets being finite. Observe also that

$$(iv) \qquad A \cap F = A_{i,i} \bigsqcup A_{i,o} \qquad \text{so that} \qquad |A \cap F| = |A_{i,i}| + |A_{i,o}|$$

$$(v) \qquad \gamma \ \text{ induces a bijection } \ A_{i,i} \bigsqcup \gamma^{-1}(A_{o,i}) \to \gamma(A) \cap F$$
$$\text{so that} \qquad |\gamma(A) \cap F| = |A_{i,i}| + |A_{o,i}|$$

$$(vi) \qquad \partial_{\mathcal{R}} F \supset \partial_{\{\gamma,\gamma^{-1}\}} F \supset \gamma(A_{i,o}) \cup A_{o,i}$$
$$\text{so that} \qquad |A_{i,o}| + |A_{o,i}| \leq 2|\partial_{\mathcal{R}} F| \leq 2\epsilon|F|.$$

It follows from (iv) to (vi) that

$$(vii) \qquad \big| |\gamma(A) \cap F| - |A \cap F| \big| \leq 2\epsilon|F|.$$

Using the definition of the mean μ_F and the conclusion of the Følner condition, one may rewrite (vii) as

$$(viii) \qquad \big| \mu_F\big(\gamma(A)\big) - \mu_F(A) \big| \leq 2\epsilon$$

so that one obtains finally

$$\big| \mu\big(\gamma(A)\big) - \mu(A) \big| \leq 2\delta + 2\epsilon \leq 4\delta$$

using (ii), (iii) and $(viii)$. As the choice of δ is arbitrary, this ends the proof of one implication of Theorem 25. □

27. Remark. In case of a locally finite graph X with finitely many orbits of vertices under the full automorphism group (for example in case of a Cayley graph), Følner condition is equivalent to the existence of a nested sequence $F_1 \subset F_2 \subset \ldots$ of finite subsets of the vertex set X^0 such that $\cup_{n \geq 1} F_n = X^0$ and $\lim_{n\to\infty} |\partial F_n|/|F_n| = 0$; see our Section III.2 for amenable graphs and for the notation ∂F_n, and Theorem 4.39 in [Soa] for the equivalence.

In the case of a group G acting on a set X, the Følner condition is most often expressed in a way involving the symmetric difference between a finite subset F of X and its image gF by some $g \in G$; for the equivalence of this with the analogue of our Definition 23, see Proposition 4.3 in [Ros1].

For groups, Følner condition implies the existence of Følner sets with extra tiling properties, and this is useful for showing extensions to amenable groups of the Rohlin theorem from ergodic theory [OrWe].

III. Amenability and paradoxical decompositions for metric spaces

III.1. Gromov condition and doubling condition

Let X be a metric space and let d denote the distance on X.

For $S, T \subset X$, a mapping $\phi : S \to T$ (*not* necessarily a bijection) is a *bounded perturbation of the identity* if $\sup_{x \in S} d(\phi(x), x) < \infty$. We will denote by

$$\mathcal{B}(X)$$

the collection of all these maps. (This would be an example of a "pseudo-semi-group", but we will not use this term again below.)

As in Example 2.iii, we denote by $\mathcal{W}(X)$ the pseudogroup of all bijections, between subsets of X, which are bounded perturbations of the identity.

For a subset A of X and a real number $k > 0$, we denote by

$$\mathcal{N}_k(A) \,=\, \{\, x \in X \mid d(x, A) \leq k \,\}$$

the *k-neighbourhood* of A in X.

Recall that a metric space is *discrete* if its subsets of finite diameter are finite.

28. Definitions. A discrete metric space X is said to be *amenable* [respectively *paradoxical*] if the pseudogroup $\mathcal{W}(X)$ is amenable [resp. paradoxical].

Caution. This definition is not convenient for non-discrete metric spaces, because the pseudogroup $\mathcal{W}(\mathbb{R})$ is paradoxical. Indeed, the bijections

$$\gamma_{even} \,:\, \bigcup_{n \in \mathbb{Z}} [2n, 2n+1[\longrightarrow \mathbb{R} \qquad \text{and} \qquad \gamma_{odd} \,:\, \bigcup_{n \in \mathbb{Z}} [2n+1, 2n+2[\longrightarrow \mathbb{R}$$

defined by $\gamma_{even}(t) = \frac{t}{2}$ and $\gamma_{odd}(t) = \frac{t}{2}$ are in $\mathcal{W}(\mathbb{R})$ and define a paradoxical decomposition of $\mathbb{R}$.

A notion of amenability for *some* non-discrete metric spaces is suggested in Remark 42.

29. Definition. A discrete metric space X is said to satisfy the *Gromov condition* if there exists a mapping $\phi : X \to X$ in $\mathcal{B}(X)$ such that

$$\left|\phi^{-1}(x)\right| \geq 2$$

for all $x \in X$.

This terminology refers in particular to the "lemme 6.17" in [GrLP], introduced there as "le meilleur moyen de montrer qu'un groupe est non-moyennable"; see also Item 0.5.C_1'' in [Gro3].

30. Definition. The discrete metric space X satisfies the *doubling condition* if there exists a constant $K > 0$ such that

$$\left|\mathcal{N}_K(F)\right| \,\geq\, 2|F|$$

for any non-empty finite subset F of X.

It is of course equivalent to ask that there exists a constant $k > 0$ and a number $\epsilon > 0$ such that

$$\left|\mathcal{N}_k(F)\right| \geq (1+\epsilon)|F|$$

for any non-empty finite subset F of X; indeed, this implies $|\mathcal{N}_K(F)| \geq 2|F|$ for any non-empty finite subset F of X, with $K = nk$ and n an integer such that $(1+\epsilon)^n \geq 2$.

31. Bipartite graphs and matchings. Let $B = Bip(Y, Z; E)$ be a *bipartite* graph with two classes Y, Z of vertices and with edge set E; by definition of "bipartite", any edge $e \in E$ is incident with one vertex in Y and one vertex in Z; we consider here simple graphs, namely graphs without loops and *without multiple edges.* Recall that, for integers $k, l \geq 1$, a *perfect* (k, l)*-matching* of B is a subset M of E such that any $y \in Y$ [respectively any $z \in Z$] is incident to exactly k edges in M [resp. l edges in M].

For a set F of vertices of B, we denote by $\partial_E F$ the set of vertices in B which are not in F, and are connected to some vertex of F by some $e \in E$.

Let again X be a metric space, as earlier in the present section. With two subsets $S, T \subset X$ and a real number $K \geq 0$, one associates the bipartite graph $B_K(S, T)$ with vertex classes S and T, and with an edge connecting $x \in S$ and $y \in T$ whenever $d(x, y) \leq K$. Observe that X is discrete if and only if $B_K(X, X)$ is locally finite for all $K \geq 0$.

32. Theorem. *For a discrete metric space X, the following conditions are equivalent (with $\mathcal{B}(X)$ as before Definition 28).*

(i) *The space X is paradoxical.*

(ii) *There exists a mapping $\phi : X \to X$ in $\mathcal{B}(X)$ such that $\left|\phi^{-1}(x)\right| = 2$ for all $x \in X$.*

(iii) *There exists a mapping $\phi : X \to X$ in $\mathcal{B}(X)$ such that $\left|\phi^{-1}(x)\right| \geq 2$ for all $x \in X$ (namely X satisfies the Gromov condition).*

(iv) *The space X satisfies the doubling condition.*

(v) *There exists a real number $K > 0$ for which the bipartite graph $B_K(X, X)$ has a perfect* $(2, 1)$*-matching.*

(vi) *The pseudogroup $\mathcal{W}(X)$ does not satisfy the Følner condition.*

33. Observations. As there are amenable groups of exponential growth, for example finitely generated solvable groups which are not virtually nilpotent, Conditions (ii) and (iii) are not connected to growth, as suggested in [DeSS], but indeed to amenability, as already observed in our Introduction.

For a recent survey on growth and related matters, see [GriH].

Some of the implications of Ttheorem 32 may be made more precise. See for example Proposition 54 below.

34. Proof of Theorem 32.

(i) $\Longleftrightarrow$ (ii). If X is paradoxical, there exists a partition $X = X_1 \sqcup X_2$ and two bijections $\gamma_j : X_j \to X$ in $\mathcal{W}(X)$. The mapping $\phi : X \to X$ defined by $\phi(x) = \gamma_j(x)$ for $x \in X_j$ ($j = 1, 2$) satisfies (ii).

Conversely, given a mapping $\phi : X \to X$ as in (ii), one uses the axiom of choice to order the two points of $\phi^{-1}(x)$ for each $x \in X$, say as $\phi^{-1}(x) = \left(\gamma_1^{-1}(x), \gamma_2^{-1}(x)\right)$. This provides a paradoxical decomposition involving the mappings γ_1 and γ_2.

The implications $(ii) \Longrightarrow (iii) \Longrightarrow (iv)$ are straightforward. Condition (v) is nothing but a rephrasing of Condition (ii).

$(vi) \Longrightarrow (iv)$. If $\mathcal{W}(X)$ does not satisfy the Følner condition, there exists $\epsilon > 0$ and a non-empty finite subset $\mathcal{R}$ of $\mathcal{W}(X)$ such that, for any non-empty finite subset F of X, one has $|F \cup \partial_{\mathcal{R}} F| \geq (1+\epsilon)|F|$. Setting

$$C \ = \ \max_{\rho \in \mathcal{R} \cup \mathcal{R}^{-1}} \ \sup_{x \in \alpha(\rho)} \ d(\rho(x), x)$$

(see Definition 1 for the notation $\alpha(\rho)$), one has a fortiori

$$\left| \mathcal{N}_C(F) \right| \ \geq \ (1+\epsilon)|F|$$

for any non-empty finite subset F of X.

$(i) \Longrightarrow (vi)$. The contraposition not$(vi) \Longrightarrow$ not(i) may be checked as follows : if the pseudogroup $\mathcal{W}(X)$ satisfies Følner condition, it is amenable by Proof 26, so that $\mathcal{W}(X)$ is not paradoxical by the straightforward part of the Tarski alternative (Remark 6.i).

We have now shown all but the right lowest $\Downarrow$ in the following diagram:

$$\begin{array}{ccccccc}
 & & (v) & & & & (vi) \\
 & & \Updownarrow & & & & \Downarrow \\
(i) & \Longleftrightarrow & (ii) & \Longrightarrow & (iii) & \Longrightarrow & (iv) \\
\Downarrow & & & & & & \Downarrow \\
(vi) & & & & & & (v)
\end{array}$$

For the last implication $(iv) \Longrightarrow (v)$, we follow [DeSS] and call upon a form of the Hall-Rado Theorem. More precisely, with the notation of Theorem 35 below and with $k = K$, (iv) implies that $\left|\partial_E F\right| \geq 2|F|$ for any subset F of Y or of Z, so that (v) follows. □

All what we will need about the Hall-Rado theorem can be found in [Mir] but, as a first background, we recommend also the discussion in § III.2 of [Bol]. (Recall that "Hall" refers to *Philip* Hall.)

35. Theorem (Hall-Rado). *Let $B = Bip(Y, Z; E)$ be a locally finite bipartite graph and let $k \geq 1$ be an integer. Assume that one has*

$$\begin{aligned}
\left|\partial_E F\right| \ &\geq \ k|F| \qquad \textit{for all finite subsets } F \textit{ of } Y \\
\left|\partial_E F\right| \ &\geq \ \ |F| \qquad \textit{for all finite subsets } F \textit{ of } Z.
\end{aligned}$$

Then there exists a perfect $(k,1)$-matching of B.

On the proof. Consider the bipartite graph $B_k = B\left(\sqcup_{1 \le j \le k} Y_j, Z; E_k\right)$ where $\sqcup_{1 \le j \le k} Y_j$ denotes a disjoint union of k copies of Y, and where, for each edge $e \in E$ with ends $y \in Y$ and $z \in Z$, there is one edge $e_j \in E_k$ with ends the vertex $y_j \in Y_j$ corresponding to y and the vertex z, this for each $j \in \{1 \dots k\}$.

One one hand, the hypothesis implies that

$$\left|\partial_{E_k} F\right| \ \ge \ |F|$$

for all finite subset F of $\sqcup_{1 \le j \le k} Y_j$ or of Z. On the other hand, there exists a perfect $(k, 1)$-matchings of B if and only if there exists a perfect $(1, 1)$-matchings of B_k. It follows that one may assume $k = 1$ without loss of generality.

By the most usual form of the Hall-Rado theorem, there are subsets M_Y, M_Z of E such that the edges in M_Y [respectively in M_Z] are pairwise disjoint, and such that each $y \in Y$ [resp. each $z \in Z$] is incident with exactly one edge in M_Y [resp. in M_Z]; see, e.g., Theorem 4.2.1 in [Mir]. Thus $M_Y \cup M_Z$ define a spanning subgraph of B whose connected components are either edges, or simple polygons with a number of edges which is even and at least 4, or infinite lines. (This argument is standard : see e.g. the middle of page 317 in [Nas].)

One may color the edges of the latter subgraph in black and white such that each vertex of B is incident to exactly one black edge. The set of black edges thus obtained is a perfect $(1, 1)$-matching of B. □

If $k = 1$, observe that the condition of the Theorem is also necessary for the existence of a perfect $(1, 1)$-matching. If $k \ge 2$, it is not so (consider a complete bipartite graph with $|Y| = 1$ and $|Z| = k$), despite the statement following Definition 6 of [DSS].

36. End of proof of Theorem 25. We show here the implication "existence of an invariant mean ⇒ Følner condition", or rather its contraposition: we assume that $(\mathcal{G}, X)$ does not satisfy the Følner condition, and we have to prove that X has no $\mathcal{G}$-invariant mean.

First case: X is a metric space and $\mathcal{G}$ is the pseudogroup $\mathcal{W}(X)$. Implication $(vi) \Longrightarrow (i)$ of Theorem 32 shows that X is paradoxical, hence that X is not amenable. The proof of Theorem 25 is complete in this case.

General case. If $(\mathcal{G}, X)$ does not satisfy the Følner condition, there exists a number $\epsilon > 0$ and a non-empty finite subset $\mathcal{R}$ of $\mathcal{G}$ such that

$$|\partial_{\mathcal{R}} F| \ > \ \epsilon |F|$$

for any non-empty finite subset F of X. Define a metric $d_{\mathcal{R}}$ on X by

$$d_{\mathcal{R}}(x, y) \ = \ \min \left\{ n \in \mathbb{N} \ \middle| \ \begin{array}{l} \text{there exists} \quad \rho_1, \dots, \rho_n \in \mathcal{R} \cup \mathcal{R}^{-1} \quad \text{such that} \\ \rho_n\Big(\rho_{n-1}\big(\dots \rho_1(x) \dots\big)\Big) \text{ is defined and is equal to } y \end{array} \right\}$$

with the understanding that $d_{\mathcal{R}}(x, y) = \infty$ if there exists no such n. One has a posteriori

$$|\mathcal{N}_1(F)| \ \ge \ (1 + \epsilon)|F|$$

for any non-empty finite subset F of X, where the neighbourhood $\mathcal{N}_1(F)$ refers to the metric $d_{\mathcal{R}}$ (for the definition of $\mathcal{N}_1$, see before Definition 28). Hence the pseudogroup $\mathcal{W}(X, d_{\mathcal{R}})$ is not amenable by the previous case. As $\mathcal{W}(X, d_{\mathcal{R}}) \subset \mathcal{G}$, the pseudogroup $\mathcal{G}$ itself is not amenable either. □

37. Definition. Recall that two metric spaces X, Y are *quasi-isometric* if there exist constants $\lambda \geq 1$, $C \geq 0$ and a mapping $\phi : X \to Y$ such that

$$\frac{1}{\lambda} d(x_1, x_2) - C \ \leq \ d\big(\phi(x_1), \phi(x_2)\big) \ \leq \ \lambda d(x_1, x_2) + C$$

for all $x_1, x_2 \in X$ and

$$d\big(y, \phi(X)\big) \ \leq \ C$$

for all $y \in Y$.

Recall also that X and Y are *Lipschitz equivalent* if there exists a constant $\lambda \geq 1$ and a *bijection* $\psi : X \to Y$ such that

$$\frac{1}{\lambda} d(x_1, x_2) \ \leq \ d\big(\psi(x_1), \psi(x_2)\big) \ \leq \ \lambda d(x_1, x_2)$$

for all $x_1, x_2 \in X$. (See also Item 0.2.C in [Gro3].)

38. Proposition. *Let X and Y be two discrete metric spaces which are quasi-isometric. Then X is amenable [respectively is paradoxical] if and only if Y is so.*

Proof. The Gromov condition of Definition 29 is clearly invariant by quasi-isometry. □

39. Examples. For each prime p, there are uncountably many 2-generated p-groups which are amenable and pairwise *not* quasi-isometric; see [Gri2] for $p = 2$ and [Gri3] for $p \geq 2$.

40. Examples. There are uncountably many 2-generated torsion-free groups which are paradoxical and pairwise *not* quasi-isometric [Bow].

41. Remark. It is a result due independently to Volodymyr Nekrashevych [Nek1] and Kevin Whyte [Why] that two non-amenable discrete metric spaces X and Y are quasi-isometric if and only if they are Lipschitz equivalent. This answers a question of Gromov (Item $1.A'$ in [Gro3]); see also [Pap] and [Bogp] for partial answers.

42. Remark. Let (Ω, d_Ω) be a metric space. A subset X of Ω is a *separated net* if there exists a constant $r > 0$ for which the two following properties hold : (i) $d_\Omega(x, y) \geq r$ for all $x, y \in X$, $x \neq y$, and (ii) X is a maximal subset of Ω for this property (this implies $d_\Omega(\omega, X) \leq 2r$ for all $\omega \in \Omega$). Such nets exist by Zorn's Lemma.

If the metric space Ω is "slim and well-behaved" in the sense of [MaMT], for example if Ω is a Riemannian manifold with Ricci curvature bounded from below and the injectivity radius of the exponential map positive, then two nets in Ω are quasi-isometric to each other. (See Theorems 3.3 and 3.4 in [MaMT], as well as [Kan1], [Kan2] and [Nek1], [Nek2].) For such slim and well-behaved spaces, there are natural notions of amenability and paradoxes, defined via their nets; this has appeared in several places, including [BlWe]. Proposition 38 carries over to these spaces, by definition.

43. Examples. There are uncountably many Riemann surfaces of constant curvature -1 which are amenable as metric spaces, and which are pairwise *not* quasi-isometric [Gri5].

III.2. Graphs as metric spaces, isoperimetric constants

Let $X = (X^0, X^1)$ be a graph with vertex set X^0 and with edge set X^1 (say X has no loops and no multiple edges, for simplicity). If X is connected, X^0 is naturally a metric space, the distance $d(x, y)$ between two vertices $x, y \in X^0$ being the minimal number of edges in a path between them.

The metric space X^0 is then discrete if and only if the graph X is locally finite.

For a disconnected graph X, there are also notions of combinatorial distances. For example, if X is a subgraph of a connected graph Y which is clear from the context, one may restrict to X^0 the distance defined on Y^0 as above. One may also set $d(x, x') = \infty$ for x, x' in different connected components of X.

44. Definition. A locally finite graph X is said to be *amenable* or *paradoxical* if the metric space X^0 is so in the sense of Definition 28.

For a subset F of X^0, the boundary $\partial_E F$ defined in graph theoretical terms in Item 31 (here $E = X^1$) coincides with $\mathcal{N}_1(F) \setminus F$, where $\mathcal{N}_1 F$ is the neighbourhood defined in metrical terms before Definition 28. We will write

$$\partial F = \mathcal{N}_1(F) \setminus F$$

below.

45. Definition. The *isoperimetric constant* of the graph X is

$$\iota(X) = \inf \left\{ \frac{|\partial F|}{|F|} \;\middle|\; F \subset X^0 \quad \text{is finite and non-empty} \right\}.$$

For example, $\iota(X) = 0$ as soon as X has a finite connected component.

46. Variations. There are several variations on the definition of the isoperimetric constant in the literature, because a boundary ∂F could be defined using

either vertices *outside* F as here (before Definition 45) or in [BeSc] and [McM1],
or vertices *inside* F as in [Dod] or [CoSa],
or vertices *both* inside and outside F as in [OrWe, page 24],
or *edges* connecting vertices inside F to those outside F as in [BiMS] or [Kai1].

For example, denoting by $\partial_* F$ the set of *edges* connecting a vertex of F to a vertex outside F, there is another isoperimetric constant

$$\iota_*(X) \; = \; \inf \left\{ \frac{|\partial_* F|}{|F|} \,\middle|\, F \subset X^0 \quad \text{is finite and non-empty} \right\}.$$

for the graph X. One has $\iota_*(X) \geq \iota(X)$; if X has maximal degree k, one has also $\iota_*(X) \leq k\iota(X)$.

47. Example. *Let d be an integer, $d \geq 3$. For a tree T in which every vertex is of degree at least d, the isoperimetric constant satisfies the inequality*

$$\iota(T) \; \geq \; d-2.$$

If T is regular of degree d, then $\iota(T) = d-2$.

Proof. As we have not found a convenient published reference for this very standard fact, we indicate now a proof.

Let F be a finite subset of the vertex set of $T(d)$, let X denote the subgraph of $T(d)$ induced by F, let $X_1, \ldots, X_N$ denote its connected components, and let F_i denote the vertex set of X_i, for $i \in \{1, \ldots, N\}$. We claim that

$$|\partial F| \; \geq \; (d-2)|F| + 2.$$

Assume first that X is connected. We proceed by induction on $|F|$. If $|F| = 1$, then $|\partial F| \geq d = (d-2)|F| + 2$ and the claim is obvious. Assume now that $|F| = k \geq 2$; let $y \in F$ be a vertex of X-degree 1, and let Y be the subgraph of X induced by $F \setminus \{y\}$. One has

$$|\partial F| \; \geq \; |\partial(F \setminus \{y\})| + d - 2 \; \stackrel{*}{\geq} \; (d-2)\big(|F|-1\big) + 2 + d - 2 \; = \; (d-2)|F| + 2$$

where $\stackrel{*}{\geq}$ holds because of the induction hypothesis. (It is easy to check that $|\partial F| = (d-2)|F| + 2$ in case X is a *regular* tree of degree d.)

Assume now that X has $N \geq 2$ connected components, and proceed by induction on N. As T is a tree, one may assume the numerotation of the F_i 's such that ∂F_1 has at most one vertex in common with $\partial \left(\bigcup_{2 \leq i \leq N} F_i \right)$. Then

$$|\partial F| \; \geq \; |\partial F_1| + \Big|\partial\Big(\bigcup_{2 \leq i \leq N} F_i \Big)\Big| - 1 \; \stackrel{**}{\geq} \; (d-2)|F_1| + 2 + (d-2) \sum_{i=2}^{N} |F_i| + 2 - 1 \; > \; (d-2)|F| + 2$$

where $\stackrel{**}{\geq}$ holds because of the induction hypothesis.

It follows that $\iota(T) \geq d-2$, with equality for a d-regular tree. □

Recall that a *hanging chain* of length k in a graph X is a path of length k (with $k+1$ vertices, $k-1$ so-called inner ones and the two end-vertices) with all inner vertices of degree 2 in X. It is obvious that, if X has hanging chains of arbitrarily large lengths, then $\iota(X) = 0$. The following is a kind of converse, for trees.

48. Example. *Let T be a connected infinite locally finite tree without end-vertices and let k be an integer, $k \geq 2$. If T has no hanging chain of length $> k$, then*

$$\iota(T) \geq \frac{1}{2k}.$$

Also $\iota(T) = 0$ if and only if T has arbitrary long hanging chains.

Proof : see the proof of Corollary 4.2 in [DeSS]. □

Other interesting estimates of isoperimetric constants appear, for example, in § 4 of [McM1].

49. Definitions. On a locally finite graph X, there is a natural *simple random walk* with corresponding *Markov operator* T. Suppose for simplicity that X is connected and of bounded degree. Consider the Hilbert space $\ell^2(X^0, deg)$ of functions h from X^0 to $\mathbb{C}$ such that $\sum_{x \in X^0} deg(x)|h(x)|^2 < \infty$, and the bounded self-adjoint operator T defined on this Hilbert space by

$$(Th)(x) = \frac{1}{deg(x)} \sum_{y \sim x} h(y)$$

for $h \in \ell^2(X^0, deg)$, $x \in X^0$, where $y \sim x$ indicates a summation over the neighbours y of the vertex x. The *spectral radius* of X is

$$\begin{aligned} \rho(X) &= \sup \left\{ \langle h|Th\rangle \,\middle|\, h \in \ell^2(X)\,,\ \|h\|_2 \leq 1 \right\} \\ &= \sup \left\{ |\lambda| \;\middle|\; \lambda \ \text{ is in the spectrum of } \ T \right\}. \end{aligned}$$

Observe that $1 - T$ is a natural analogue on X of a Laplacian, so that $1 - \rho(X)$ is often referred to as the first eigenvalue of the Laplacian or (more appropriately) as the bottom of its spectrum.

It is also known that, for a real number λ, the following are equivalent :

(i) there exists $F : X^0 \to]0, \infty[$ such that $\displaystyle\frac{1}{deg(x)} \sum_{y \sim x} F(y) = \lambda F(x)$,

(ii) there exists $F : X^0 \to]0, \infty[$ such that $\displaystyle\frac{1}{deg(x)} \sum_{y \sim x} F(y) \leq \lambda F(x)$,

(iii) one has $\lambda \geq \rho(X)$,

so that (i) and (ii) indicate alternative definitions of the spectral radius. In terms of the Laplace operator, (i) and (ii) are respectively conditions about $(1 - \lambda)$-harmonic and $(1 - \lambda)$-superharmonic functions. (For a proof in terms of graphs, see Proposition 1.5 in [DoKa]. But there are earlier proofs in the literature on irreducible stationary discrete Markov chains. The equivalence of (ii) and (iii) is standard; the equivalence with (i) is more delicate : [Harr] and [Pru].)

For $x, y \in X^0$ and for an integer $n \geq 0$, denote by $p^{(n)}(x,y)$ the probability that a simple random walk starting at x is at y after n steps. Then one has also $\rho(X) = \limsup_{n\to\infty} \sqrt[n]{p^{(n)}(x,y)}$; in particular, the value of this lim sup is independent on x and y. From this probabilistic interpretation of $\rho(X)$, one deduces easily that, for a connected graph X which is regular of degree $d \geq 2$, one has $\rho(X) \geq 2\sqrt{d-1}/d$; equality holds if and only if X is a tree.

(More generally, for any transition kernel $p : X^0 \times X^0 \to [0,\infty[$ with reversible measure $\mu : X^0 \to]0,\infty[$, so that $\sum_{z\in X^0} p(x,z) = 1$ and $\mu(x)p(x,y) = p(y,x)\mu(y)$ for all $x, y \in X^0$, one introduces the Hilbert space $\ell^2(X^0,\mu)$, and the self-adjoint operator T defined by the kernel p on $\ell^2(X^0,\mu)$. Then the norm of T is again equal to $\limsup_{n\to\infty} \sqrt[n]{p^{(n)}(x,y)}$.)

50. Lemma (an isoperimetric inequality). *For a graph X which is regular of degree $d \geq 2$, one has*

$$\iota(X) \geq 4\,\frac{1-\rho(X)}{\rho(X)}.$$

Proof. Let $\mathbb{X}^1$ denote the set of oriented edges of X. (If X is finite, the cardinal of $\mathbb{X}^1$ is twice the number of geometric edges of X.) Each $e \in \mathbb{X}^1$ has a head $e_+ \in X^0$ and a tail $e_- \in X^0$. For a function $h \in \ell^2(X^0, deg)$ with real values, one has

$$\langle h|Th\rangle = \sum_{x\in X^0} h(x)\sum_{y\sim x} h(y) = \sum_{e\in\mathbb{X}^1} h(e_+)h(e_-) = \|h\|^2 - \frac{1}{2}\sum_{e\in\mathbb{X}^1}\Big(h(e_+)-h(e_-)\Big)^2.$$

Let now F be a finite non-empty subset of X^0, with boundary ∂F. Consider the function $h \in \ell^2(X^0, deg)$ defined by

$$h(x) = \begin{cases} \dfrac{1}{\sqrt{d}} & \text{if} \quad x \in F \\ \dfrac{1}{2\sqrt{d}} & \text{if} \quad x \in \partial F \\ 0 & \text{otherwise} \end{cases}$$

One has clearly

(*) $$\|h\|^2 = |F| + \frac{1}{4}|\partial F| \geq |F|\left(1 + \frac{\iota(X)}{4}\right).$$

One has also

$$\frac{1}{2}\sum_{e\in\mathbb{X}^1}\Big(h(e_+)-h(e_-)\Big)^2 = \sum_{y\in\partial F}\sum_{x\sim y}\Big(h(y)-h(x)\Big)^2 \leq |\partial F|\,d\,\frac{1}{4d}.$$

Together with (*), this implies that

$$\rho(X) \geq \frac{\langle h|Th\rangle}{\|h\|^2} \geq 1 - \frac{|\partial F|}{4|F|\left(1+\frac{\iota(X)}{4}\right)}.$$

Taking the infimum over $\frac{|\partial F|}{|F|}$ one obtains

$$\rho(X) \geq 1 - \frac{\frac{\iota(X)}{4}}{1 + \frac{\iota(X)}{4}}$$

and the lemma follows. □

The previous lemma appears in several places (see N^o 51 below). It is related to Theorem 3.1 of [BiMS], which is stated in terms of the constant $\iota_*(X)$ of our Item 46, and which shows that $\iota_*(X) \geq 4(1-\rho(X))$. Recently, T. Smirnova-Nagnibeda has improved the latter to

$$\iota_*(X) \geq \frac{d^2}{d-1}(1 - \rho(X))$$

(the improvement comes from choosing a test-function, playing the role of the function h in the proof above, which is more efficient than the one chosen in [BiMS]).

For a *majoration* of $\iota(X)$ in terms of $1 - \rho(X)$ and d (namely for an analogue of the "Cheeger's inequality"), see Theorem 2.3 in [Dod] or Theorem 3.2 in [BiMS] (in each case with normalizations different from ours).

51. Theorem. *Let X be a connected graph which is of bounded degree. The following are equivalent :*

(i)	X *is paradoxical*	*(see Definition 44),*
(ii)	$\iota(X) > 0$	*(see Definition 45),*
(iii)	$\rho(X) < 1$	*(see Definition 49),*
(iv)	$p^{(n)}(x,y) = o(\sigma^n)$	*for some* $\sigma \in]0,1[$ *and for all* $x, y \in X^0$

and they imply that

(v) *the simple random walk on X is transient.*

On the proof. The equivalence $(i) \Longleftrightarrow (ii)$ is a reformulation of Theorem 25 on the Følner condition.

The equivalence $(ii) \Longleftrightarrow (iii)$ may be viewed as a discrete analogue of the Cheeger-Buser inequalities for Riemannian manifolds [Che], [Bus]. For graphs as in the present theorem, it can be found in [Dod], [Var], [DoKe], [DoKa], [Ger], [Anc], [Kai1]; there are also similar arguments showing appropriate estimates for *finite* graphs in several papers by Alon et alii, quoted in [Lub] (in particular near Propositions 4.2.4 and 4.2.5).

For $(iii) \Longleftrightarrow (iv)$ and for equivalence with other conditions, see Theorem 4.27 in Soardi's notes on Networks [Soa].

The implication $(iii) \Longrightarrow (v)$ is obvious.

For groups, the equivalence

$$\text{amenability} \qquad \Longleftrightarrow \qquad \rho(X) = 1$$

goes back to the pioneering papers of Kesten [Kes1], [Kes2]. See also [Day3] and the review in [Woe]. □

There are other conditions equivalent to (i) to (iv) above, for example in terms of norms of Markov operators on ℓ^p-spaces; see [Kai1].

For locally finite graphs which are not necessarily of bounded degree, one has to modify some of the definitions above. Thus, for a finite set F of vertices of a graph X, one considers the sum $\|F\|$ of the degrees of the vertices in F, the number $\|\partial F\|$ of edges with one end in F and the other end outside F, and the infimum $\tilde{\iota}(X)$ of the quotients $\|\partial F\| \,/\, \|F\|$ (compare with Definition 45). For graphs of bounded degree, one has $\tilde{\iota}(X) = 0 \Longleftrightarrow \iota(X) = 0$, but in general on may have $\tilde{\iota}(X) = 0$ and $\iota(X) > 0$[2]. By a particular case of a result of Kaimanovich (Theorem 5.1 in [Kai1]), one has $\tilde{\iota}(X) > 0 \Longleftrightarrow \rho(X) < 1$.

Graphs of unbounded degree are also covered by the arguments in [DoKa] and [DoKe].

Graphs give rise to several kinds of algebras, and it is a natural question in each case to ask how the properties of Theorem 51 translate. For Gromov's *translation algebras* (see the end of 8.C_2 in [Gro3]), there is a hint in [Ele1]. For other algebras associated with graphs (and more generally with oriented graphs), see [KPRR] and [KPR]. Amenable properties of certain kind of graphs (more precisely of bipartite graphs with appropriate weights) are also important in the study of subfactors; see various works by S. Popa, including [Pop1] and [Pop2].

Amenability has of course been one of the most important notions in the theory of operator algebras since the works of von Neumann. We will not discuss more of this here, but only refer to [Co2] and [Hel].

IV. Estimates of Tarski numbers

IV.1. From relative growth to Tarski number of paradoxical decompositions

Let G be a finitely generated group, given as a quotient

$$\pi \,:\, F_m \longrightarrow G$$

[2]Here is an example shown to us by Vadim Kaimanovich. Let $(h_j)_{j\geq 1}$ be a sequence of integers, all at least 2, and consider first a rooted tree Y in which a vertex at distance n of the root is of degree

$$\begin{cases} k+2 & \text{if} \quad n = \sum_{j=1}^{k} h_j \quad \text{for some} \quad k \geq 1, \\ 3 & \text{otherwise.} \end{cases}$$

Consider then the graph X obtained from Y by adding, for each vertex x of Y at distance $n = \sum_{j=1}^{k} h_j$ from the root (for some k), the $\frac{1}{2}(k+1)(k+2)$ edges between the successors of x in Y. Then one has $\iota(X) > 0$ (because Y is a spanning tree for X) and $\tilde{\iota}(X) = 0$ (because X contains induced subgraphs which are complete graphs on $k+2$ vertices for k arbitrarily large). One has also $\rho(X) = 1$.

of the free group F_m on m generators $s_1, \ldots, s_m$, for some $m \geq 1$. The purpose of the present section is to review notions which will be used in IV.2.

52. Recall: relative growth, spectral radius and isoperimetric constant. Let $\ell : F_m \to \mathbb{N}$ denote the word length on F_m with respect to $s_1, \ldots, s_m$. For each integer $k \geq 0$, let $\sigma(Ker(\pi), k)$ denote the cardinality of the set $\{ w \in Ker(\pi) \mid \ell(w) = k \}$. The *relative growth* of $Ker(\pi)$ (some authors say "the cogrowth of G"!) is, by definition,

$$\alpha_{Ker(\pi)} = \limsup_{k\to\infty} \sqrt[k]{\sigma(Ker(\pi), k)}.$$

It is easy to check that $\sqrt{2m-1} \leq \alpha_{Ker(\pi)} \leq 2m-1$. Still for $Ker(\pi) \neq \{1\}$, one shows more precisely that $\sqrt{2m-1} < \alpha_{Ker(\pi)}$ [Gri1].

The corresponding Cayley graph (with vertex set G and with an edge between two vertices x, y if and only if $\ell(xy^{-1}) = 1$) has a spectral radius given by the formula

$$\rho = \begin{cases} \dfrac{\sqrt{2m-1}}{m} & \text{if} \quad 1 \leq \alpha \leq \sqrt{2m-1} \\ \dfrac{\sqrt{2m-1}}{2m}\left(\dfrac{\sqrt{2m-1}}{\alpha} + \dfrac{\alpha}{\sqrt{2m-1}}\right) & \text{if} \quad \sqrt{2m-1} < \alpha \leq 2m-1 \end{cases}$$

[Gri1]. It follows that the three conditions

$$\begin{aligned} \alpha &= 2m-1 \\ \rho &= 1 \\ G &\quad \text{is amenable} \end{aligned}$$

are equivalent; the equivalence of the last two is due to Kesten, as already recalled in the proof of Theorem 51. (In the present setting for the formula giving ρ as a function of α, one has $1 \leq \alpha \leq \sqrt{2m-1}$ if and only if $\alpha = 1$, if and only if $Ker(\pi) = \{1\}$; but the formula makes sense and is correct for subgroups of F_m which need not be normal, and then the range $1 \leq \alpha \leq \sqrt{2m-1}$ is meaningful.)

53. Isoperimetric constant and doubling characteristic distance. Let X be a graph, with its set X^0 of vertices viewed as a metric space for the combinatorial distance d as in Section III.2. A *doubling characteristic distance* for X is (if it exists) an integer K for which the doubling condition of Definition 30 holds, namely an integer K such that

$$|\mathcal{N}_K(F)| \geq 2|F|$$

for any non-empty finite subset F of X^0. If the isoperimetric constant $\iota(X)$ of Definition 45 is strictly positive, the integer

$$K_X = \left\lceil \frac{\log 2}{\log(1 + \iota(X))} \right\rceil$$

is clearly a doubling characteristic distance, where $\lceil t \rceil$ indicates the least integer larger than or equal to t.

54. Proposition. *Let X be a graph with isoperimetric constant $\iota(X) > 0$; define K_X as in the previous number. Then there exists a paradoxical decomposition involving a partition $X^0 = X_1^0 \sqcup X_2^0$ and two bounded perturbations of the identity $\phi_i : X_j^0 \to X^0$ in $\mathcal{W}(X^0)$ such that*

$$sup_{x \in X_j^0} d(\phi_j(x), x) \leq K_X \qquad (j = 1, 2).$$

Proof : this is a quantitative phrasing of the implication $(iv) \Longrightarrow (i)$ of Theorem 32, and follows from our Proof 34. □

55. Four functions. Let m be an integer, $m \geq 2$.

For $\alpha \in]\sqrt{2m-1}, 2m-1]$, set $\rho = \frac{\sqrt{2m-1}}{2m}\left(\frac{\sqrt{2m-1}}{\alpha} + \frac{\alpha}{\sqrt{2m-1}}\right) \in \left]\frac{\sqrt{2m-1}}{m}, 1\right]$.

For $\rho \in]0, 1]$, set $\iota(\rho) = 4\frac{1-\rho}{\rho} \in [0, \infty[$.

For $\iota \in [0, \infty[$, set $K(\iota) = \left\lceil \frac{\log 2}{\log(1+\iota)} \right\rceil \in \{1, 2, 3, \ldots, \infty\}$ (with $\lceil \ldots \rceil$ as in 53).

For $K \in \{1, 2, 3, \ldots, \infty\}$, set $b_m(K) = \frac{m(2m-1)^K - 1}{m-1}$.

Observe that $\alpha \mapsto \rho_m(\alpha)$ and $K \mapsto b_m(K)$ are increasing, while $\rho \mapsto \iota(\rho)$ and $\iota \mapsto K(\iota)$ are decreasing. Observe also that, in the Cayley graph of a group G with respect to a set of m generators, a ball of radius K has at most $b_m(K)$ elements, and precisely $b_m(K)$ elements in case G is free on m generators.

56. Theorem. *Let $G = F_m/N$ be a group given as a quotient of the free group on m generators by a normal subgroup $N \neq \{1\}$. Let α_G denote the corresponding relative growth and let $\iota(X)$ denote the isoperimetric constant of the corresponding Cayley graph X (see Definition 45 and Item 52). Using the notations of the previous number, one has :*

(i) if $\alpha_G \leq \alpha$ for some $\alpha \leq 2m-1$, the Tarski number of G satisfies

$$\mathcal{T}(G) \leq 2b_m\Big(K\Big(\iota\left(\rho_m(\alpha)\right)\Big)\Big),$$

(ii) if $\iota(X) \geq \iota$ for some $\iota \geq 0$, then

$$\mathcal{T}(G) \leq 2b_m\big(K(\iota)\big).$$

Proof. For (i), one has $\iota(X) \geq \iota\left(\rho_m(\alpha)\right)$ by the formula of Item 52 and by the isoperimetric inequality of Lemma 50, and this implies $K_X \leq K\left(\iota\left(\rho_m(\alpha)\right)\right)$. If $\phi_j : X_j^0 \to X^0$ are as in Proposition 54, one may write X_j^0 as a finite disjoint union of the sets

$$A_{j,g} = \left\{x \in X_j^0 \mid \phi_j(x) = gx\right\}$$

for g in the ball $B^G(K_X) = \{g \in G \mid \ell(g) \leq K_X\}$ (compare with Observation 9), this for $j = 1$ and $j = 2$. As $|B^G(K_X)| \leq b_m(K_X)$, this ends the proof of (i). The end of the argument shows also (ii). □

57. Comments and examples. Observe that we have argued with the Cayley graph of G related to the *right*-invariant distance $d(x, y) = \ell\left(xy^{-1}\right)$ on G, so that the *left*-multiplications $x \mapsto gx$ are bounded perturbations of the identity.

Let us now test the inequalities of Theorem 56.

(i) Let F_2 denote the free group of rank 2 and let X denote the Cayley graph of F_2 with respect to some free basis (X is of course a regular tree of degree 4). Kesten [Kes1] has computed the spectral value of the corresponding simple random walk as $\rho(X) = \frac{\sqrt{3}}{2} \approx 0.86603$ so that $\iota(X) \geq 4\frac{1-\rho(X)}{\rho(X)} \approx 0.6188$. Hence $K_X = \left\lceil \frac{\log 2}{\log(1.6188)} \right\rceil = 2$ is a doubling characteristic distance. The resulting estimate

$$\mathcal{T}(F_2) \leq 2|B^{F_2}(2)| = 2\big(2{\cdot}3^2 - 1\big) = 34$$

compares rather badly with the correct value $\mathcal{T}(F_2) = 4$.

A similar computation with the Cayley graph Y of F_3 with respect to a free basis gives $\rho(Y) = \frac{\sqrt{5}}{3} \approx 0.7454$, so that $\iota(Y) \geq 1.366$. Hence $K = 1$ is a doubling characteristic distance. Consequently $\mathcal{T}(F_3) \leq 2|B^{F_3}(1)| = 14$. As F_3 is a subgroup of F_2 one may improve the previous estimate to

$$\mathcal{T}(F_2) \leq 14$$

by Observation 19.

(ii) Consider again the Cayley graph X of F_2. Its isoperimetric constant is precisely $\iota(X) = deg(X) - 2 = 2$ by Example 47. Hence $K_X = \left\lceil \frac{\log 2}{\log 3} \right\rceil = 1$ is a doubling characteristic distance; thus

$$\mathcal{T}(F_2) \leq 2|B^{F_2}(1)| = 10,$$

which compares better than the previous estimate with $\mathcal{T}(F_2) = 4$.

These computations indicate that some effort should be given to sharpen the isoperimetric inequality of Lemma 50 used above (see Question 62.a).

IV.2. Tarski number for Ol'shanskii groups and for Burnside groups

58. On Ol'shanskii groups. We consider first a family of groups investigated in [Ol1]. (See also [Ol2] both for this family and for other ones, discovered by the same author, and relevant for the subject discussed here.) For any $\epsilon > 0$, there exists one of these groups given as a quotient $\pi : F_2 \twoheadrightarrow G$ for which the relative growth α_G satisfies

$$\sqrt{3} < \alpha_G \leq \sqrt{3} + \epsilon$$

and which is consequently non-amenable. Moreover Ol'shanskii has shown that these groups do not have any non-abelian free subgroups; thus their Tarski number satisfy $\mathcal{T}(G) \geq 5$, and $\mathcal{T}(G) \geq 6$ in case of torsion groups (Proposition 21). From the relative growth extimate above and from Theorem 56 (see also the first computation of Item 57), one obtains the following.

59. Proposition. *There exists a two-generator non-amenable torsion-free group G without non-abelian free subgroup, for which the Tarski number $\mathcal{T}$ satisfies*

$$5 \leq \mathcal{T}(G) \leq 34.$$

There exist a two-generator non-amenable torsion group G, with all proper subgroups cyclic, for which

$$6 \leq \mathcal{T}(G) \leq 34.$$

(The constructions of these groups are due to Ol'shanskii.)

60. On Burnside groups. We consider next the Burnside group $B(m,n)$, given as the quotient of the free group F_m of rank $m \geq 2$ by the normal subgroup generated by $\{x^n\}_{x \in F_m}$, for an odd integer $n \geq 665$. It is obvious that $B(m,n)$ does not contain any free group not reduced to $\{1\}$. It is known that $B(m,n)$ is infinite, indeed of exponential growth (see VI.2.16 in [Ady1]), and indeed not amenable [Ady2].

From Theorem 3 and the last but one line in [Ady2][3], one has the relative growth estimate

$$\alpha \leq (2m-1)^{\frac{1}{2}+\frac{1}{15}+\frac{5.69}{57}}$$

where $\frac{1}{2}+\frac{1}{15}+\frac{5.69}{57}$ is strictly smaller than, but near, $\frac{2}{3}$.

For $m=2$, Theorem 56 shows that one has successively $\alpha < \sqrt[3]{9}$, hence $\rho < \frac{\sqrt{3}}{4}\left(\frac{\sqrt{3}}{\sqrt[3]{9}} + \frac{\sqrt[3]{9}}{\sqrt{3}}\right) \approx 0.881$, hence $\iota(X) \geq 4\frac{1-\rho(X)}{\rho(X)} \approx 0.540$, hence $K = \left\lceil \frac{\log 2}{\log(1.540)} \right\rceil = 2$, hence finally

$$\mathcal{T}(B(2,n)) \leq 2|B^{F_2}(2)| = 2\big(2 \cdot 3^2 - 1\big) = 34.$$

For $m=3$, the corresponding computations give $\alpha < \sqrt[3]{25}$, hence $\rho < \frac{\sqrt{5}}{6}\left(\frac{\sqrt{5}}{\sqrt[3]{25}} + \frac{\sqrt[3]{25}}{\sqrt{5}}\right) \approx 0.772$, hence $\iota \geq 1.181$, hence $K=1$, hence finally

$$\mathcal{T}(B(3,n)) \leq 2|B^{F_3}(1)| = 14.$$

Let m_1, m_2 be such that $2 \leq m_1 \leq m_2 \leq \infty$ and let n be as above. It follows from general principles on relatively free groups in varieties of groups that $B(m_2,n)$ has a subgroup isomorphic to $B(m_1,n)$; see [NeuH], Statements 12.62 and 13.41. It is also known that $B(m_1,n)$ has a subgroup isomorphic to $B(m_2,n)$; see [Sir], and also § 35.2 in [Ol2]. Thus, it follows from Observation 10 that one has $\mathcal{T}\big(B(m,n)\big) = \mathcal{T}\big(B(3,n)\big)$ for any $m \geq 2$.

This and Proposition 21 show the following.

[3]There are printing mistakes in the English version of [Ady2]. In Theorem 3 of this paper, first the C should read G, and second the exponent of $(2m-1)$ should read

$$\left[\frac{1}{2} + \frac{\beta}{\gamma_R} + \frac{4}{\delta_R}\left(log_{2m-1}\left(e\Big(1+\frac{\delta_R}{4\gamma_R}\Big)\right)\right)\right]$$

(with the largest parenthesis () as above). Also, in the last but one line of the paper, $\frac{1}{15}+\frac{6}{57}$ should be replaced by $\frac{1}{15}+\frac{5.69}{57}$, which is indeed a number strictly smaller than $\frac{1}{6}$!

61. Theorem. *For $m \geq 2$ and for n odd and at least 665, the Tarski number of the Burnside group $B(m,n)$ satisfies*

$$6 \leq \mathcal{T}\big(B(m,n)\big) \leq 14.$$

Let us mention that it is unknown whether, for n large, $B(m,n)$ has infinite amenable quotients. (A question of Stepin, which is Problem 9.7 of [Kou].) Similarly one could ask what are the Tarski numbers of non-amenable quotients of these groups.

62. Questions of continuity.

Question (a) : given $\epsilon > 0$, does there exist $\delta > 0$ such that, for any quotient group G of a free group F with spectral radius satisfying $\rho(G) < \rho(F) + \delta$, one has necessarily an estimate $\iota(G) > \iota(F) - \epsilon$ for the isoperimetric constants ? More generally, can one sharpen the inequality $\iota(X) \geq 4\frac{1-\rho(X)}{\rho(X)}$ of Lemma 50 ?

Question (b) : given $\delta > 0$, does there exist $\eta > 0$ such that, for any quotient group G of a free group F with exponential growth rate satisfying $\omega(G) > \omega(F) - \eta$, one has necessarily an estimate $\rho(G) < \rho(F) + \delta$?

(For $\omega(G)$, see [GriH]. If the free group F above is of rank m and is considered together with a free basis, recall that $\omega(F) = 2m - 1$, $\rho(F) = \sqrt{2m-1}$, and $\iota(F) = 2m - 2$. The coefficients $\omega(G)$, $\rho(G)$ and $\iota(G)$ are of course taken with respect to the images in G of free generators in F.)

Assume the two questions above have affirmative answers; then : (i) for a convenient group G of Ol'shanskii, $\iota(G) \geq 2 - \epsilon$, and $K = 1$, and consequently $\mathcal{T}(G) \leq 10$; (ii) for the Burnside groups $B(2,n)$ of Theorem 61 with n large enough, one would have $\omega(G) \geq 3 - \epsilon$ (VI.2.16 in [Ady1]), and $K = 1$, and consequently also $\mathcal{T}\big(B(m,n)\big) = \mathcal{T}\big(B(2,n)\big) \leq 10$ for any $m \geq 2$ and n odd large enough.

V. Superamenability

V.1. Superamenability and subexponential growth

63. Definition. *A pseudogroup $(\mathcal{G}, X)$ is superamenable if the pseudogroup $\big(\mathcal{G}_{(A)}, A\big)$ defined in Example 2.iv is amenable for any nonempty subset A of X.*

In case of a pseudogroup $\mathcal{W}(X)$, Remark 3.vi shows that one may read this definition in two ways. More precisely, a discrete metric space X is superamenable if, for any subspace A of X, one has

(i) the metric space A is amenable, i.e. the pseudogroup $\mathcal{W}(A)$ is amenable,

or equivalently

(ii) the restriction $\mathcal{W}(X)_{(A)}$ of the pseudogroup $\mathcal{W}(X)$ to A is amenable.

Observe that superamenability of discrete metric spaces is invariant by quasi-isometry, because of Proposition 38.

A finitely generated group is *superamenable* if it so as a metric space, for the combinatorial distance on its Cayley graph with respect to a finite generating set (this definition of superamenability does not depend on the choice of the generating set).

This notion, due to Rosenblatt [Ros2], carries over to not necessarily finitely generated groups, and indeed to topological groups, but we will not use this below.

64. Definition Let X be a discrete metric space; for a point $x \in X$ and a number $r \geq 0$, we denote by $\beta_x^X(r)$ the cardinality of the closed ball of radius r around x in X. The space X is of

$$\textit{subexponential growth} \text{ if } \qquad \limsup_{r\to\infty} \sqrt[r]{\beta_x^X(r)} = 1$$

$$\textit{exponential growth} \text{ if } \qquad 1 < \limsup_{r\to\infty} \sqrt[r]{\beta_x^X(r)} < \infty$$

$$\textit{superexponential growth} \text{ if } \qquad \limsup_{r\to\infty} \sqrt[r]{\beta_x^X(r)} = \infty.$$

(Observe[4] that any of these holds for some $x \in X$ if and only if it holds for all $x \in X$, and also if and only if it holds for any pair (X', x') with X' quasi-isometric to X. In particular, subexponential growth and exponential growth make sense for finitely generated groups, without any mention of a generating set.)

65. Lemma. *Inside a metric space of subexponential growth, any subspace is also of subexponential growth.*

Proof. For a subspace Y of a space X, one may choose in the previous definition the point x inside Y. Then the lemma follows from the obvious inequality $\beta_x^Y(r) \leq \beta_x^X(r)$, for all $r \geq 0$. □

For historical perspective, let us recall that a simple argument going back to [AdVS] shows that a finitely generated group which is of subexponential growth is amenable, and indeed superamenable (Theorem 4.6 in [Ros2]).

As a consequence, one has $\iota(X) = 0$ for any Cayley graph X of a finitely generated group of subexponential growth. There are further connections between growth and isoperimetry, due to Varopoulos and others. More precisely, consider for example a finitely generated group G generated by a finite set S, the corresponding growth function β_S^G defined by

$$\beta_S^G(n) = |\{ g \in G \mid \text{ the } S\text{-word length of } g \text{ is at most } n \ \}$$

for all $n \geq 0$, and the *isoperimetric profile* I_S^G defined by

$$I_S^G(n) = \max_{m \leq n} \min_{F \subset X^0,\, |F|=m} |\partial F|$$

for all $n \geq 1$, where X^0 denotes the vertex set of the Cayley graph of G with respect to S (namely $X^0 = G$!); then, for various classes of groups, there are quite precise estimates relating the growth function β_S^G and the isoperimetric profile I_S^G; see in particular [CoSa] and [PiSa].

In our context, the argument of [AdVS] provides the following result.

[4] Unlike in some other places of this paper (such as Proof 36), we insist here that the distance between two points of X is always *finite*.

66. Theorem. *A discrete metric space of subexponential growth is superamenable.*

Proof. Let X be a discrete metric space of subexponential growth. By the previous lemma, it is enough to show that X is amenable; we will show that X satisfies the Følner condition.

Consider a finite subset $\mathcal{R}$ in the pseudogroup $\mathcal{W}(X)$, a point $x_0 \in X$ and a number $\epsilon > 0$. Set

$$C = \left\lceil \max_{\rho \in \mathcal{R} \cup \mathcal{R}^{-1}} \sup_{x \in \alpha(\rho)} d(x, \rho(x)) \right\rceil .$$

As

$$\limsup_{r \to \infty} \sqrt[r]{\beta^X_{x_0}(r)} = 1$$

there exists a strictly increasing sequence of integers $(r_k)_{k \geq 1}$ such that

$$\lim_{k \to \infty} \frac{\beta^X_{x_0}(r_k + C)}{\beta^X_{x_0}(r_k)} = 1.$$

Set

$$F_k = \text{ball of radius } \quad r_k \quad \text{centered at } \quad x_0 \quad \text{in} \quad X$$

for all $k \geq 1$.

As $\partial_{\mathcal{R}} F_k \subset \mathcal{N}_C F_k \setminus F_k$ for all $k \geq 1$, one has

$$\lim_{k \to \infty} \frac{|\partial_{\mathcal{R}} F_k|}{|F_k|} = 0$$

so that $(F_k)_{k \geq 1}$ is a "Følner sequence" (see Definition 23), and this ends the proof. □

The following criterium for graphs will be used in Section V.2. Recall that a metric space X is *long-range connected* if there is a constant $C > 0$ such that every two points x and y in X can be joined by a finite chain of points

$$x_0 = x \, , \, x_1 \, , \, \ldots \, , \, x_n = y$$

such that

$$d(x_{i-1}, x_i) \leq C$$

for all $i \in \{1, \ldots, n\}$ (see Item 0.2-A_2 in [Gro3]).

67. Proposition. *A connected locally finite graph is superamenable if and only if all its long-range connected subgraphs are amenable.*

Proof of the non-trivial implication. Given a graph X which is *not* superamenable, we have to show that there exists a long-range connected subset Z of its vertex set X^0 which is not amenable (as a metric space, for the combinatorial distance of X).

By hypothesis, there exists a subset Y of X^0 and a mapping $\phi : Y \to Y$ such that $\sup_{y \in Y} d(\phi(y), y) \leq C$ for some constant $C \geq 0$, and such that $|\phi^{-1}(y)| \geq 2$ for all $y \in Y$. Set $Z = \mathcal{N}_C(Y)$, and let $(Z_i)_{i \in I}$ be an enumeration of the connected components of Z. For all $i \in I$, set $Y_i = Y \cap Z_i$. As ϕ is a C-bounded perturbation of the identity, one has $\phi^{-1}(Y_i) \subset Z_i$, and it follows that $\phi^{-1}(Y_i) \subset Y_i$, for all $i \in I$. Hence Y_i is paradoxical for each $i \in I$. □

V.2. Examples with trees

Let $\mathcal{S}_2$ denote the free *semi*-group on two generators. From the natural word length, one defines on $\mathcal{S}_2$ a metric making it a discrete metric space which is of exponential growth, and indeed paradoxical. Thus, any finitely generated group containing a sub-semi-group isomorphic to $\mathcal{S}_2$ has a paradoxical subspace (the group being viewed as a metric space), and consequently is not superamenable.

68. Question. Does there exist a finitely generated group which is amenable, not superamenable, and without sub-semi-group isomorphic to $\mathcal{S}_2$?

This question is due to Rosenblatt, who conjectured the answer to be negative (see [Ros2], just after Theorem 4.6 and after Corollary 4.20); he also observed the following alternative for a finitely generated solvable group : either the group has a nilpotent subgroup of finite index, and then the group is superamenable, or the group contains S_2 as a sub-semi-group, and then the group is not superamenable (Theorems 4.7 and 4.12 in [Ros2]).

However, Question 68 has been answered positively by the second author as follows.

69. Examples [Gri4]**.** *For each prime p, there exist uncountably many finitely generated p-groups which are*

- *of exponential growth,*
- *without any sub-semi-group isomorphic to* $\mathcal{S}_2$,
- *amenable,*
- *not superamenable.*

On the proof. This involves wreath products[5] $G = C_p \wr H$, where C_p denote a cyclic group of order p and where H is one of the p-groups of intermediate growth constructed in [Gri2-3]. To show that G is not superamenable, the method is to construct a paradoxical tree in an appropriate Cayley graph of G.

As a torsion group, G does not contain $\mathcal{S}_2$.

The two other claims are straightforward. □

70. Question. Does there exist a finitely generated group which is superamenable and of exponential growth ?

This question, formulated as Item 12.9.a and Problem C.12 of [Wag], is still open.

One way to make the question more precise is recorded as Problem 16.11 in the *Kourovka Notebook* [Kou] : does there exist a finitely generated semi-group S with cancellation having subexponential growth and such that the group of left quotients $G = S^{-1}S$ has exponential growth ? (The group of quotients would exist, because the so-called "Ore condition" holds; see for example § 1.10 and 12.4 in [ClPr].) The point is that such a

[5]In the English translation of [Gri4], the Russian word for "wreath product" has been incorrectly translated as "amalgamated product" !

semi-group of subexponential growth is superamenable and that a group of quotients of a superamenable semi-group is a superamenable group.

Here is however a straightforward construction.

71. Example. *There exists a discrete metric space which is of superexponential growth and which is superamenable.*

Proof. Consider a sequence $(d_k)_{k\geq 0}$ of integers ≥ 2 and a sequence $(h_k)_{k\geq 1}$ of integers ≥ 1. Let X be a rooted tree in which a vertex at distance n of the root is of degree

$$\begin{cases} d_k & \text{if} \quad n = \sum_{j=1}^{k} h_j \\ 2 & \text{otherwise} \end{cases}$$

(given the two sequences, this completely defines the tree up to isomorphism).

If $\liminf_{k\to\infty} h_k = \infty$, a long-range connected subspace Y of the vertex set of X cannot satisfy the Gromov condition (compare with Proposition 35 above, i.e. with Corollary 4.2 of [DeSS]). It follows from Proposition 67 that X is superamenable.

Now the growth sequence of X with respect to the base point satisfies

$$\beta^X(n+1) \geq \prod_{j=0}^{k} d_j \qquad \text{for} \qquad n = \sum_{j=1}^{k} h_j,$$

so that, if the sequence $(d_k)_{k\geq 0}$ is increasing rapidly enough, one has

$$\limsup_{m\to\infty} \sqrt[m]{\beta^X(m)} = \infty$$

and X is of superexponential growth. For example, if $d_j = \left(\sum_{i=1}^{j} h_i\right)!$, then

$$\beta^X(n+1) \geq d_k = n!$$

whenever $n = \sum_{j=1}^{k} h_j$, and this implies $\limsup_{m\to\infty} \sqrt[m]{\beta^X(m)} = \infty$ by Stirling's formula. □

72. Variation on the previous example. *There exists a graph of bounded degree which is of exponential growth and which is superamenable.*

Proof. Consider a rooted tree X in which a vertex at distance n of the root is of degree

2 if $(k-1)k \leq n < k^2$ for some $k \geq 1$,

3 if $k^2 \leq n < k(k+1)$ for some $k \geq 1$.

The growth function of X with respect to the root satisfies

$$2^{\frac{k(k+1)}{2}} \leq \beta^X\big(k(k+1)\big) \leq 3^{\frac{k(k+1)}{2}}$$

for all $k \geq 1$, so that X is clearly of exponential growth. Example 48 implies that X is superamenable. □

73. Question. Let G and H be two finitely generated groups which are superamenable; is the product $G \times H$ superamenable ?

This question appears in [Ros2] (just before Proposition 4.21), and the answer is still unknown. Here is however an example, for which we are grateful to Laurent Bartholdi.

74. Example. *There exist two superamenable discrete metric spaces X, Y such that the direct product $X \times Y$ is not superamenable, for the metric defined by*

$$d_{X\times Y}\big((x_1, y_1), (x_2, y_2)\big) \ = \ d_X(x_1, x_2) + d_Y(y_1, y_2).$$

Proof. Let $(h_k)_{k\geq 1}$ be a strictly increasing sequence of integers ≥ 1. Let X be a rooted tree in which a vertex at distance n of the root is of degree

$$\begin{cases} 3 & \text{if} \quad \sum_{j=0}^{2k} h_j \leq n < \sum_{j=1}^{2k+1} h_j \quad \text{for some} \quad k \geq 0, \\ 2 & \text{otherwise} \end{cases}$$

(with $\sum_{j=0}^{2k} h_j = 0$ for $k = 0$). And let Y be a rooted tree in which a vertex at distance n of the root is of degree

$$\begin{cases} 3 & \text{if} \quad \sum_{j=1}^{2k+1} h_j \leq n < \sum_{j=1}^{2k+2} h_j \quad \text{for some} \quad k \geq 0, \\ 2 & \text{otherwise.} \end{cases}$$

Observe that both X and Y are superamenable, because each of their infinite connected subgraphs has arbitrarily large hanging chains. Observe also that, for each integer n, there is either in X or in Y a vertex of degree 3 at distance n of the relevant root. It follows that the product of the two metric spaces defined by X and Y, for the distance $d_{X\times Y}$ defined above, contains a paradoxical tree. Consequently, $X \times Y$ is not superamenable. □

75. Paradoxical subtrees in paradoxical graphs. It is known that a paradoxical graph contains a paradoxical tree [BeSc]. It is unknown whether a connected paradoxical graph necessarily contains a paradoxical tree which is *spanning,* i.e. which contains all vertices of the original graph (this is Problem 2 in § 4 of [DeSS]).

However, Benjamini and Schramm have shown that, if X is a paradoxical graph with $\iota(X) \geq n$ for some integer $n \geq 2$, then X has a *spanning forest* of which every connected component is a tree with one vertex of degree $n-1$ and all other vertices of degree $n+1$. This implies that X has a paradoxical *spanning tree.*

76. A conjecture of V. Trofimov. This appears as Problem 12.87 in the Kourovka Notebook [Kou]. Let X be a connected undirected graph without loops and multiple edges and suppose that its automorphism group $Aut(X)$ acts transitively on the vertices. Is it true that one of the following holds ?

(i) the stabilizer of a vertex of X is finite,

(ii) the action of $Aut(X)$ on the vertices of X admits a non-trivial imprimitivity system σ with finite blocks for which the stabilizer of a vertex of the factor-graph X/σ in $Aut(X/\sigma)$ is finite,

(iii) there exists a natural number n such that the graph, obtained from X by adding edges connecting distinct vertices the distance between which in X is at most n, contains a tree all of whose vertices have valence 3.

If the answer to this question was positive, this would imply that a graph of subexponential growth having a transitive group of automorphisms is essentially a Cayley graph of a group.

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