M147, Fall 2009, Final   Name______________________

Note for Fall 2010. This is the final exam for M147, Fall 2009, and the exam this year will have the same format (ten multiple choice questions and five problems for which work is to be written out). This exam covered one section that will not be on the final exam for Fall 2010, Section 6.3. In light of this, you should skip problems 9, 10, and 15. On the other hand, you should go back and work problems 7, 8, 9, 10, 14b, and 15 from last year's Exam 3. (We did not cover the material for Problem 14a this year.) Solutions are given at the end of the exam.

Calculators are not allowed on the exam. The first ten problems are multiple choice. Written out work on these problems will not be checked, so take care in marking your answers. For Problems 11-15 unjustified answers will not receive credit. You may need the following summation formulas.

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}; \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}; \qquad \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

1. [5 pts] Evaluate the limit

$$\lim_{x\to -2^-} \frac{x^2}{x+2}.$$

(a) $-\infty$

(b) $+\infty$

(c) 1

(d) -1

(e) None of the above

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2. [5 pts] Evaluate the limit

$$\lim_{x\to\infty} (1+\frac{1}{\sqrt{x}})^x.$$

(a) $e^{\frac{1}{2}}$

(b) $e^{-\frac{1}{2}}$

(c) $-\infty$

(d) $+\infty$

(e) None of the above

3. [5 pts] Find a value for c so that the given function is differentiable at $x = 0$.

$$f(x) = \begin{cases} 2\sin 3x, & x < 0 \\ \sin cx, & x \geq 0 \end{cases}.$$

(a) 0

(b) 2

(c) 3

(d) 6

(e) None of the above

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4. [5 pts] Compute the derivative of the function

$$f(x) = \frac{\sqrt{x}\sin x}{x^2 + 1}.$$

(a) $\frac{\sin x}{2\sqrt{x}(x^2+1)} - \frac{\sqrt{x}\cos x}{(x^2+1)} - \frac{2x^{3/2}\sin x}{(x^2+1)^2}$

(b) $\frac{\sin x}{2\sqrt{x}(x^2+1)} + \frac{\sqrt{x}\cos x}{(x^2+1)} - \frac{2x^{3/2}\sin x}{(x^2+1)^2}$

(c) $\frac{\sin x}{2\sqrt{x}(x^2+1)} + \frac{\sqrt{x}\cos x}{(x^2+1)} + \frac{2x^{3/2}\sin x}{(x^2+1)^2}$

(d) $\frac{\sin x}{2\sqrt{x}(x^2+1)} - \frac{\sqrt{x}\cos x}{(x^2+1)} + \frac{2x^{3/2}\sin x}{(x^2+1)^2}$

(e) None of the above

5. [5 pts] Compute the derivative of the function

$$f(x) = \sqrt{1 - \sin^2(e^x)}.$$

(a) $-\frac{\sin(e^x)\cos(e^x)}{\sqrt{1-\sin^2(e^x)}}$

(b) $\frac{\sin(e^x)\cos(e^x)}{\sqrt{1-\sin^2(e^x)}}$

(c) $-\frac{e^x\sin(e^x)\cos(e^x)}{\sqrt{1-\sin^2(e^x)}}$

(d) $\frac{e^x\sin(e^x)\cos(e^x)}{\sqrt{1-\sin^2(e^x)}}$

(e) None of the above

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6. [5 pts] Find an equation for the line that is tangent to the curve described by

$$x^2 + y^2 = xy + 1,$$

at the point $(1, 0)$.

(a) $y = -2(x - 1)$

(b) $y = 2(x - 1)$

(c) $y = -\frac{1}{2}(x - 1)$

(d) $y = \frac{1}{2}(x - 1)$

(e) None of the above

7. [5 pts] Compute the derivative

$$\frac{d}{dx}\int_{\sin x}^{\cos x} \sqrt{1+y^3}dy.$$

(a) $(\sqrt{1+\cos^3 x} - \sqrt{1+\sin^3 x})(\cos x - \sin x)$

(b) $-3\sin x\cos^2 x\sqrt{1+\cos^3 x} - 3\cos x\sin^2 x\sqrt{1+\sin^3 x}$

(c) $-\sqrt{1+\cos^3 x}\sin x - \sqrt{1+\sin^3 x}\cos x$

(d) $-\sqrt{1+\cos^3 x}\cos x - \sqrt{1+\sin^3 x}\sin x$

(e) None of the above

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8. [5 pts] Evaluate the indefinite integral

$$\int \frac{\cos(\sqrt{x})}{\sqrt{x}}dx.$$

(a) $2\sin(\sqrt{x}) + C$

(b) $-2\sin(\sqrt{x}) + C$

(c) $\frac{1}{2}\sin(\sqrt{x}) + C$

(d) $-\frac{1}{2}\sin(\sqrt{x}) + C$

(e) None of the above

9. [5 pts] Find the area of the region bounded by the graphs of $y = x^2 + 2$ and $y = x - 1$ for $x \in [-1, 2]$.

(a) $\frac{17}{2}$

(b) 9

(c) $\frac{19}{2}$

(d) $\frac{21}{2}$

(e) None of the above

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10. [5 pts] Suppose the population at time t of a certain species is modeled by the equation

$$p(t) = 8 + 2\sin(2\pi t),$$

where t is measured in years. Find the average population during the first half of the year, $t \in [0, \frac{1}{2}]$.

(a) $8 - \frac{4}{\pi}$

(b) $8 + \frac{4}{\pi}$

(c) $1 - \frac{1}{\pi}$

(d) $1 + \frac{1}{\pi}$

(e) None of the above

11. [10 pts] (Part (c) of this problem is on the next page.) Let

$$f(x) = \frac{x^2}{x-2}, \quad x \in [-2, 6].$$

11a. Locate the critical points of f and determine the intervals on which f is increasing and the intervals on which f is decreasing.

11b. Locate the possible inflection points for f and determine the intervals on which f is concave up and the intervals on which it is concave down.

11c. Evaluate f at the critical points, the possible inflection points, and at the boundary points and use this information to sketch a graph of this function.

12. [10 pts] A fence 8 feet tall runs parallel to a tall building at a distance 1 foot from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building?

13. [10 pts] Find all fixed points for the recursion equation

$$x_{t+1} = x_t + \frac{1}{2}x_t(1 - x_t^2),$$

and classify each as unstable or asymptotically stable.

14. [10 pts] Use the method of Riemann sums to evaluate

$$\int_{-1}^{3} 1 - x^2 dx.$$

15. [10 pts] Find the volume obtained when the region enclosed by the curves $y = x^2 + 1$ and $y = 2$ is rotated about the x-axis.

Solutions to 2009 Exam 3

7. a

8. b

9. d

10. a

14a.

$$\lim_{\|P\|\to 0}\sum_{k=1}^{n} e^{\sqrt{1+c_k^2}}\Delta x_k.$$

Here, P denotes a partition of $[1,7]$, $\Delta x = x_k - x_{k-1}$, $\|P\| = \max_{k=1,2,\ldots,n}\Delta x_k$, and $c_k \in [x_{k-1}, x_k]$ for each $k = 1, 2, \ldots n$.

14b. $f(x)$ is continuous at $x = \frac{\pi}{2}$ and clearly continuous at the other points in $[0,\pi]$, so the Fundamental Theorem of Calculus applies.

15. Using right endpoints, and intervals of equal width $\Delta x = \frac{2}{n}$, we have $x_k = 2 + k\Delta x = 2 + \frac{2k}{n}$. We compute

$$\begin{aligned} A_n &= \sum_{k=1}^{n}(2+\frac{2k}{n})^2\frac{2}{n} = \sum_{k=1}^{n}(4+\frac{8k}{n}+\frac{4k^2}{n^2})\frac{2}{n} \\ &= \frac{8}{n}\sum_{k=1}^{n}1 + \frac{16}{n^2}\sum_{k=1}^{n}k + \frac{8}{n^3}\sum_{k=1}^{n}k^2 \\ &= \frac{8}{n}n + \frac{16}{n^2}\frac{n(n+1)}{2} + \frac{8}{n^3}\frac{n(n+1)(2n+1)}{6}. \end{aligned}$$

Finally,

$$\int_2^4 x^2 dx = \lim_{n\to\infty} A_n = 8 + 8 + \frac{8}{3} = \frac{56}{3}.$$

Solutions to 2009 Final Exam

1. a

2. d

3. d

4. b

5. c

6. b

7. c

8. a

9. d

10. b

11a. We compute

$$f'(x) = \frac{x(x-4)}{(x-2)^2},$$

from which we find:

$$\begin{aligned}\text{f is increasing on } &[-2,0] \cup [4,6]\\ \text{f is decreasing on } &[0,2) \cup (2,4].\end{aligned}$$

11b. We compute

$$f''(x) = \frac{8}{(x-2)^3},$$

from which we find:

$$\begin{aligned}\text{f is concave up on } &(2,6)\\ \text{f is concave down on } &(-2,2).\end{aligned}$$

11c. The evaluations are:

$$\begin{aligned} f(-2) &= -1\\ f(0) &= 0\\ \lim_{x\to 2^-} \frac{x^2}{x-2} &= -\infty\\ \lim_{x\to 2^+} \frac{x^2}{x-2} &= +\infty\\ f(4) &= 8\\ f(6) &= 9.\end{aligned}$$

The graph is given in Figure 1.

12. Let y denote the location where the top of the ladder touches the building and let x denote the distance from the bottom of the ladder to the fence. Finally, let l denote the length of the ladder. (See Figure 2.)

By the Pythagorean Theorem we have the relationship

$$(x+1)^2 + y^2 = l^2 = L,$$

and so we want to minimize

$$L = (x+1)^2 + y^2.$$

(Recall that the values of x and y that minimize $L = l^2$ will also minimize l.) We need a relationship between x and y, and for this we observe that similar triangle gives the relationship

$$\frac{8}{x} = \frac{y}{x+1} \Rightarrow y = \frac{8}{x}(x+1).$$

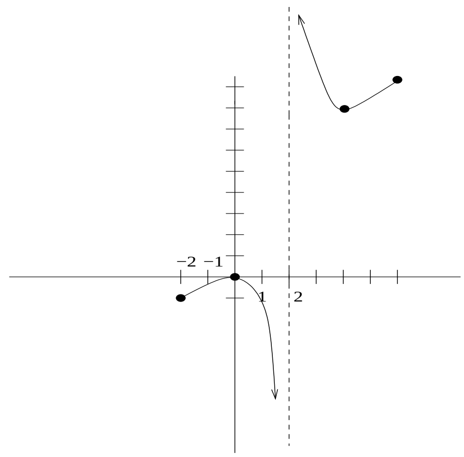

Figure 1: Figure for Problem 11 solution.

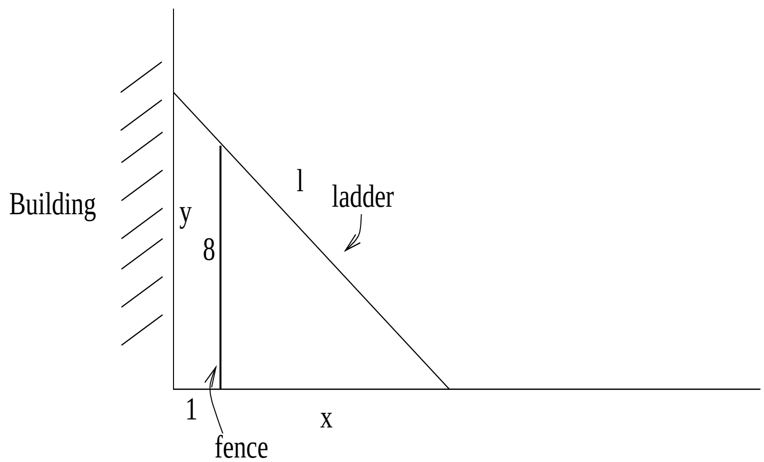

Figure 2: Figure for Problem 12.

In this way,

$$L(x) = (x+1)^2 + \frac{64}{x^2}(x+1)^2 = (x+1)^2(1+\frac{64}{x^2}),$$

with $0 < x < \infty$. To find the critical points of L, we compute

$$\begin{aligned} L'(x) &= 2(x+1)(1+\frac{64}{x^2}) + (x+1)^2(-\frac{128}{x^3}) = (x+1)\Big[2 + \frac{128}{x^2} - \frac{128}{x^2} - \frac{128}{x^3}\Big] \\ &= (x+1)\Big[2 - \frac{128}{x^3}\Big]. \end{aligned}$$

The critical points are $x = -1, 0, 4$, where $x = 4$ solves

$$2 - \frac{128}{x^3} = 0.$$

To check if this is a minimum, we compute

$$\begin{aligned} \lim_{x\to 0} L(x) &= \infty \\ L(4) &= 125 \\ \lim_{x\to\infty} L(x) &= \infty. \end{aligned}$$

Finally, the acual length of the ladder is

$$l = \sqrt{125}.$$

13. The fixed points are solutions of

$$x = x + \frac{1}{2}x(1-x^2) \Rightarrow 0 = \frac{1}{2}x(1-x^2) \Rightarrow x = -1, 0, 1.$$

In order to check stability, we set

$$f(x) = x + \frac{1}{2}x(1-x^2) = \frac{3}{2}x - \frac{1}{2}x^3,$$

and compute

$$f'(x) = \frac{3}{2} - \frac{3}{2}x^2.$$

Computing

$$\begin{aligned} f'(-1) &= 0 \\ f'(0) &= \frac{3}{2} \\ f'(1) &= 0, \end{aligned}$$

we see that -1 and 1 are stable, while 0 is unstable.

14. First, $\Delta x = \frac{4}{n}$ and $x_k = -1 + k\Delta x = -1 + \frac{4k}{n}$. Now,

$$\begin{aligned} A_n &= \sum_{k=1}^{n} \left[1 - (-1 + \frac{4k}{n})^2\right]\frac{4}{n} = \sum_{k=1}^{n} \left[1 - (1 - \frac{8k}{n} + \frac{16k^2}{n^2})\right]\frac{4}{n} \\ &= \sum_{k=1}^{n} \left[\frac{8k}{n} - \frac{16k^2}{n^2}\right]\frac{4}{n} = \sum_{k=1}^{n} \left[\frac{32k}{n^2} - \frac{64k^2}{n^3}\right] \\ &= \frac{32}{n^2}\sum_{k=1}^{n} k - \frac{64}{n^3}\sum_{k=1}^{n} k^2 = \frac{32}{n^2} \cdot \frac{n(n+1)}{2} - \frac{64}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}. \end{aligned}$$

Finally, we compute

$$\int_{-1}^{3} 1 - x^2 dx = \lim_{n\to\infty} A_n = 16 - \frac{64}{3} = -\frac{16}{3}.$$