M147 Practice Problems for Exam 1

Exam 1 will be Thursday September 30, 7:30-9:30 p.m. (The exam room will be announced in class.) It will cover sections 1.1, 1.2, 1.3, 3.1, 3.2, 3.3, 3.4, 3.5. Calculators will not be allowed on the exam. The first ten problems on the exam will be multiple choice. Work will not be checked on these problems, so you will need to take care in marking your solutions. For the remaining problems unjustified answers will not receive credit.

1. Which of the following functions corresponds with the graph in Figure 1.

(a) $y = \log_2 x$

(b) $y = \log_{\frac{1}{2}} x$

(c) $y = 2^x$

(d) $y = (\frac{1}{2})^x$

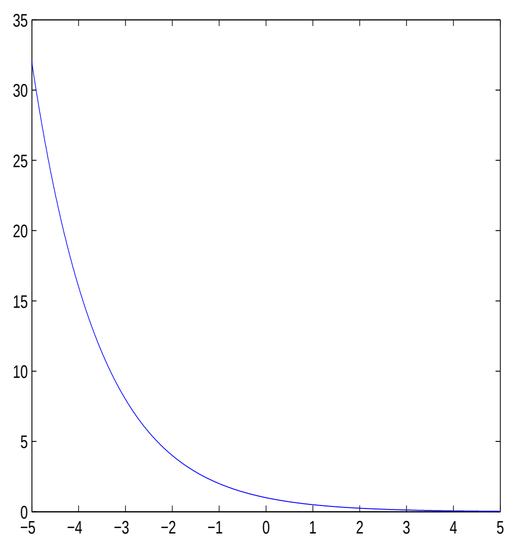

Figure 1: Figure for Problem 1.

2. Use a logarithmic transformation to find a linear relationship between (appropriate transformations of) x and y if

$$y = 2 \times 7^{4x}.$$

3. Given the semilog plot in Figure 2, find a functional relationship between x and y.

4. Given the double-log plot in Figure 3, find a functional relationship between x and y.

5a. When $\log y$ is graphed as a function of x, a straight line results. Graph the straight line, on a semilog plot, associated with the points $(1, 3)$ and $(7, 9)$, given in the original coordinates.

5b. Find a functional relationship between x and y for the situation described in (5a).

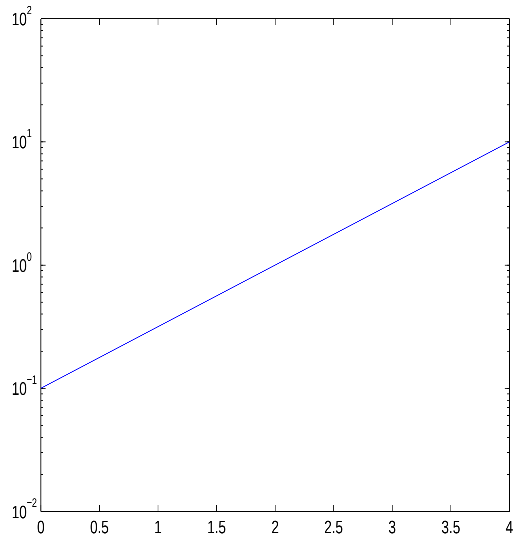

Figure 2: Figure for Problem 3.

6. Compute each of the following limits:

6a.
$$\lim_{x\to 2} \frac{x^2-4}{x-2}.$$

6b.
$$\lim_{x\to 3^-} \frac{x}{x^2-2x-3}.$$

6c.
$$\lim_{x\to 0} \frac{\sin 7x}{x}.$$

6d.
$$\lim_{x\to 1} \frac{\sqrt{x^2+1}-\sqrt{x+1}}{x-1}.$$

6e.
$$\lim_{x\to -\infty} \frac{x^3-x^2+1}{1-x^2}.$$

6f.
$$\lim_{x\to \infty} (e^{-x}\sin x).$$

6g.
$$\lim_{x\to \infty} \sqrt{x^2-x}-\sqrt{x^2+x}.$$

7. Find all points at which
$$\frac{\ln(1-x)}{\ln(1+x)}$$

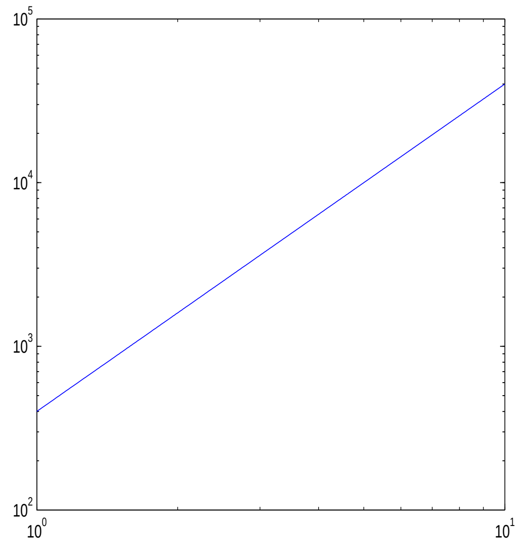

Figure 3: Figure for Problem 4.

is continuous.

8. Find a value for c that makes the given function continuous at all points.

$$f(x) = \begin{cases} x^2 + 1, & x \leq 1 \\ x - c, & x > 1 \end{cases}.$$

9a. Prove that the equation

$$e^{\sqrt{x}} - \frac{1}{1 + x^2} = 1$$

has a solution on the interval $(0, 1)$.

9b. Determine the number of steps required to approximate the solution in (9a) with an error less than .01.

10. Prove that the equation

$$e^x - 2 = \sin x$$

has at least one real-valued solution.

11. Use the bisection method to approximate a root of

$$x^4 + x^3 + x - 1 = 0$$

with a maximum error less than $\frac{1}{3}$.

12. Use the bisection method to approximate $\sqrt[4]{5}$ with a maximum error less than $\frac{1}{3}$.

13. The quadratic equation

$$x^2 - x - 1 = 0$$

has two roots, one positive and one negative. The positive root is traditionally referred to as the *golden mean*. Use the bisection method to approximate the value of this root with a maximum error less than $\frac{1}{5}$.

Solutions

1. Notice that the function is defined for negative values of x, so it cannot be either of the logarithms. Since the function is decreasing, it must be exponentiation with a base less than 1, and this leaves only (d) $y = (\frac{1}{2})^x$.

2. You can proceed by taking a logarithm of this equation to any base. Though 7 would be a reasonable base here, it is sufficiently uncommon that I'll use base 10. That is,

$$\log y = \log 2 \times 7^{4x} \Rightarrow \log y = \log 2 + 4x \log 7.$$

The linear relationship is

$$\log y = (4 \log 7)x + \log 2.$$

3. This is a semilog plot with $\log y$ on the vertical axis and x on the horizontal. That is, the line has an equation of the form

$$\log y = mx + b.$$

We can read directly from the plot that $b = -1$. (I.e., $b = \log 10^{-1} = -1$) Likewise, the slope is

$$m = \frac{\log 10^1 - \log 10^{-1}}{4 - 0} = \frac{1 - (-1)}{4} = \frac{1}{2}.$$

We have, then,

$$\log y = \frac{1}{2}x - 1.$$

In order to get a functional relationship, exponentiate each side with base 10,

$$10^{\log y} = 10^{\frac{1}{2}x - 1} \Rightarrow y = 10^{-1}(10^{\frac{1}{2}})^x.$$

4. This is a double-log plot, so we look for a relationship of the form

$$\log y = m \log x + b,$$

for which

$$y = 10^b x^m.$$

Reading the plot, we see that

$$m = \frac{\log(4 \times 10^4) - \log(4 \times 10^2)}{\log 10^1 - \log 10^0} = \frac{\log \frac{4 \times 10^4}{4 \times 10^2}}{1} = \log 10^2 = 2.$$

Likewise,

$$b = \log 400.$$

We conclude

$$y = 10^{\log 400} x^2 = 400x^2.$$

5. The slope is

$$m = \frac{\log 9 - \log 3}{7 - 1} = \frac{\log \frac{9}{3}}{6} = \frac{1}{6} \log 3 = \log 3^{\frac{1}{6}}.$$

Using the first point, we can write the line in point-slope form

$$\log y - \log 3 = \log 3^{\frac{1}{6}} (x - 1),$$

which we can rearrange as

$$\log y = x \log 3^{\frac{1}{6}} + \log 3 - \log 3^{\frac{1}{6}} = \log 3^{\frac{x}{3}} + \log \frac{3}{3^{\frac{1}{6}}} = \log 3^{\frac{x}{3}} + \log 3^{\frac{5}{6}}.$$

Finally, we take each side as an exponent of 10:

$$10^{\log y} = 10^{\log 3^{\frac{x}{3}} + \log 3^{\frac{5}{6}}} = 10^{\log 3^{\frac{x}{3}}} \cdot 10^{\log 3^{\frac{5}{6}}} = 3^{\frac{x}{3}} \cdot 3^{\frac{5}{6}}.$$

We conclude

$$y = e^{\frac{x+5}{6}}.$$

6a. Compute

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \to 2} (x + 2) = 4.$$

Notice in particular that we don't have to be able to evaluate the function at a point to compute its limit at that point.

6b. Compute

$$\lim_{x \to 3^-} \frac{x}{x^2 - 2x - 3} = \lim_{x \to 3^-} \frac{x}{(x - 3)(x + 1)} = -\infty.$$

6c. We make the substitution $y = 7x$, and our limit becomes

$$\lim_{y \to 0} \frac{\sin y}{(y/7)} = 7 \lim_{y \to 0} \frac{\sin y}{y} = 7.$$

You won't lose points on a problem like this if you omit the explicit substitution.

6d. In this case, we rationalize the numerator,

$$\begin{aligned}
\lim_{x \to 1} \frac{\sqrt{x^2 + 1} - \sqrt{x + 1}}{x - 1} &= \lim_{x \to 1} \frac{\sqrt{x^2 + 1} - \sqrt{x + 1}}{x - 1} \cdot \frac{\sqrt{x^2 + 1} + \sqrt{x + 1}}{\sqrt{x^2 + 1} + \sqrt{x + 1}} \\
&= \lim_{x \to 1} \frac{x^2 + 1 - (x + 1)}{(x - 1)(\sqrt{x^2 + 1} + \sqrt{x + 1})} \\
&= \lim_{x \to 1} \frac{x^2 - x}{(x - 1)(\sqrt{x^2 + 1} + \sqrt{x + 1})} = \lim_{x \to 1} \frac{x(x - 1)}{(x - 1)(\sqrt{x^2 + 1} + \sqrt{x + 1})} \\
&= \lim_{x \to 1} \frac{x}{(\sqrt{x^2 + 1} + \sqrt{x + 1})} = \frac{1}{2\sqrt{2}}.
\end{aligned}$$

6e. According to our rule from class, the following calculation is entirely fair:

$$\lim_{x\to-\infty}\frac{x^3-x^2+1}{1-x^2}=\lim_{x\to-\infty}\frac{x^3}{-x^2}=\lim_{x\to-\infty}(-x)=+\infty.$$

6f. Since $\sin x$ does not have a limit as $x\to\infty$ we use the Squeeze Theorem (a.k.a. the Sandwich Theorem), observing

$$-e^{-x}\le e^{-x}\sin x\le e^{-x}.$$

We have $\lim_{x\to\infty}(-e^{-x})=\lim_{x\to\infty}(e^{-x})=0$, so by the Squeeze Theorem

$$\lim_{x\to\infty}e^{-x}\sin x=0.$$

6g. We compute

$$\begin{aligned}\lim_{x\to\infty}\sqrt{x^2-x}-\sqrt{x^2+x}&=\lim_{x\to\infty}(\sqrt{x^2-x}-\sqrt{x^2+x})\cdot\frac{\sqrt{x^2-x}+\sqrt{x^2+x}}{\sqrt{x^2-x}+\sqrt{x^2+x}}\\&=\lim_{x\to\infty}\frac{(x^2-x)-(x^2+x)}{\sqrt{x^2-x}+\sqrt{x^2+x}}=\lim_{x\to\infty}\frac{-2x}{\sqrt{x^2-x}+\sqrt{x^2+x}}\cdot\frac{\frac{1}{x}}{\frac{1}{x}}\\&=\lim_{x\to\infty}\frac{-2}{\sqrt{1-\frac{1}{x}}+\sqrt{1+\frac{1}{x}}}=-1.\end{aligned}$$

7. First, observe that $\ln(1-x)$ is only defined for $x<1$ and $\ln(1+x)$ is only defined for $x>-1$, so our range is restricted to this interval. Also, we cannot divide by 0, so we must have $x\neq0$. We conclude that the points of continuity are

$$(-1,0)\cup(0,1).$$

8. We observe that the only point at which f may not be continuous is $x=1$, and at this point $f(1)=2$. In order to make the function continuous at this point, we must ensure

$$\lim_{x\to1^+}x-c=2,$$

and this requires $c=-1$.

9a. We define

$$f(x)=e^{\sqrt{x}}-\frac{1}{1+x^2}-1,$$

and compute

$$\begin{aligned}f(0)&=1-1-1=-1\\f(1)&=e-\frac{1}{2}-1>0,\end{aligned}$$

where the inequality holds because $e \approx 2.72$. We conclude from the Intermediate Value Theorem that there exists a value $c \in (0,1)$ so that

$$f(c) = 0.$$

9b. The error at step k is

$$e_k = \frac{b-a}{2^k},$$

so we require (here $a = 0$ and $b = 1$)

$$\frac{1}{2^k} < .01 = \frac{1}{100} \Rightarrow 2^k > 100.$$

Taking a natural log of both sides, we obtain

$$\ln 2^k > \ln 100 \Rightarrow k > \frac{\ln 100}{\ln 2}.$$

(It's also reasonable to work with $\log_{10} x$ or $\log_2 x$.)

10. We begin by defining the function

$$f(x) = e^x - 2 - \sin x,$$

and we note that our goal will be to show that $f(x)$ has at least one real root. First, we observe that $f(0) = -1$. Next, we observe that since $e > 2$ we know that $e^2 > 4$, so that $f(2) = e^2 - 2 - \sin 2 > 0$. We can conclude from the Intermediate Value Theorem that there is a root on the interval $(0,2)$.

11. We begin by defining the function

$$f(x) = x^4 + x^3 + x - 1,$$

and we observe that $f(0) = -1$ and $f(1) = 2$, so that we are guaranteed a root in $(0,1)$. We take

$$c_1 = \frac{1}{2} \pm \frac{1}{2},$$

and compute

$$f(\frac{1}{2}) = (\frac{1}{2})^4 + (\frac{1}{2})^3 + \frac{1}{2} - 1 = \frac{1}{16} + \frac{1}{8} + \frac{1}{2} - 1 = \frac{1+2+8-16}{16} = -\frac{5}{16} < 0.$$

We conclude that the root is on the interval $(\frac{1}{2}, 1)$, and our second approximation becomes

$$c_2 = \frac{\frac{1}{2}+1}{2} = \frac{3}{4} \pm \frac{1}{4},$$

and $\frac{1}{4} < \frac{1}{3}$, so this is a sufficient approximation. (**Note.** This equation has a second real root between -2 and -1, so it's possible to approximate that one instead, which is fine.)

12. Begin by noticing that $x = \sqrt[4]{5}$ is a root of $f(x) = x^4 - 5$. We observe that $f(1) = -4$ and $f(2) = 11$, so that we are guaranteed a root in $(1, 2)$. We take

$$c_2 = \frac{1+2}{2} = \frac{3}{2} \pm \frac{1}{2}.$$

This error is not small enough, so we proceed with another step, noting this time $f(\frac{3}{2}) = \frac{81}{16} - 5 = \frac{81-80}{16} = \frac{1}{16}$. Since this is positive we are guaranteed a root in $(1, \frac{3}{2})$. We take

$$c_2 = \frac{1+\frac{3}{2}}{2} = \frac{5}{4} \pm \frac{1}{4}.$$

Since $\frac{1}{4} < \frac{1}{3}$ $c_2 = \frac{5}{4}$ is sufficient.

13. We set

$$f(x) = x^2 - x - 1,$$

and begin by observing that $f(1) = -1$ and $f(2) = +1$, so that the root is on the interval $(1, 2)$. Our first approximation is

$$c_1 = \frac{1+2}{2} = \frac{3}{2} \pm \frac{1}{2}.$$

We compute

$$f(\frac{3}{2}) = (\frac{3}{2})^2 - \frac{3}{2} - 1 = \frac{9}{4} - \frac{5}{2} = -\frac{1}{4},$$

so the root is on $c \in (\frac{3}{2}, 2)$. Our second approximation is

$$c_2 = \frac{\frac{3}{2}+2}{2} = \frac{\frac{7}{2}}{2} = \frac{7}{4} \pm \frac{1}{4}.$$

This still isn't quite sufficient, so we compute

$$f(\frac{7}{4}) = (\frac{7}{4})^2 - \frac{7}{4} - 1 = \frac{49}{16} - \frac{11}{4} = \frac{49}{16} - \frac{44}{16} > 0,$$

so our root is on $c \in (\frac{3}{2}, \frac{7}{4})$. Our third approximation is

$$c_3 = \frac{\frac{3}{2}+\frac{7}{4}}{2} = \frac{\frac{6}{4}+\frac{7}{4}}{2} = \frac{13}{8} \pm \frac{1}{8}.$$

The error is small enough now, so this is our approximation. (By the way, $\frac{13}{8} = 1.6250$ and the golden mean, to four places, is 1.6180.)