M147 Practice Problems for Exam 2

Exam 2 will cover all sections of Chapter 4. The first ten problems on the exam will be multiple choice. Work will not be checked on these problems, so you will need to take care in marking your solutions. For the remaining problems unjustified answers will not receive credit.

You should memorize the following derivative formulas:

1. $\frac{d}{dx}x^r = rx^{r-1}$ for any real number r
2. $\frac{d}{dx}\sin x = \cos x$
3. $\frac{d}{dx}\cos x = -\sin x$
4. $\frac{d}{dx}\tan x = \sec^2 x$
5. $\frac{d}{dx}e^x = e^x$
6. $\frac{d}{dx}a^x = a^x \ln a$, $a > 0$ (which contains (5) as the case $a = e$)
7. $\frac{d}{dx}\ln x = \frac{1}{x}$
8. $\frac{d}{dx}\log_a x = \frac{1}{x \ln a}$, $a > 0$, $a \neq 0$ (which contains (7) as the case $a = e$)
9. $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$
10. $\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$
11. $\frac{d}{dx}\tan^{-1} x = \frac{1}{x^2+1}$

Also, be sure you are able to use the following rules of differentiation:

1. Product rule: $\frac{d}{dx}f(x)g(x) = f'(x)g(x) + f(x)g'(x)$
2. Quotient rule: $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}$
3. Chain rule: $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$
4. Derivative of a function inverse: $\frac{df^{-1}(x)}{dx} = \frac{1}{f'(f^{-1}(x))}$.

1. Use the definition of derivative to compute the derivative of the following function at $x = 0$.

$$f(x) = \begin{cases} x^2 \cos(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

2. Determine whether or not each of the following functions is differentiable at the point $x = 0$. In each case, explain why or why not.

2a.

$$f(x) = \begin{cases} x^2 + 1, & x \leq 0 \\ x^2 - 1, & x > 0 \end{cases}.$$

2b.

$$f(x) = \begin{cases} x^2 + 1, & x \leq 0 \\ 2x + 1, & x > 0 \end{cases}.$$

2c.

$$f(x) = x|x|.$$

3. Find an equation for the line that is tangent to the given curve at $x = 1$.

$$y = x^3 + 1.$$

Sketch a graph of the curve along with this tangent line.

4. A car moves along a straight road. Its location at time t is given by

$$s(t) = 20t^2, \quad 0 \le t \le 2,$$

where t is measured in hours and $s(t)$ is measured in kilometers.

4a. Graph $s(t)$ for $0 \le t \le 2$.

4b. Find the average velocity of the car between $t = 0$ and $t = 2$. Illustrate the average velocity on the graph of $s(t)$.

4c. Find the instantaneous velocity of the car at $t = 1$. Illustrate the instantaneous velocity on the graph of $s(t)$.

5. Find a point on the curve

$$y = x^2 + x + 1$$

whose tangent line is parallel to the line

$$y - 2 = 3(x - 1).$$

6. Compute the derivative of each of the following functions:

6a.

$$f(x) = x^{\frac{2}{3}} + x^{-7}.$$

6b.

$$f(x) = x \sin x.$$

6c.

$$f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

6d.

$$f(x) = (2x + \frac{1}{x})^2.$$

6e.

$$f(x) = \sin^{-1} \frac{5}{x^2}.$$

6f.

$$f(x) = \tan^{-1} \sqrt{1 - x^4}.$$

7. Suppose

$$h(x) = f(x)e^{g(x)},$$

and $f(2) = 4$, $f'(2) = 7$, $g(2) = 0$, and $g'(2) = 3$. Compute $h'(2)$.

8. Compute $f''(x)$ if

$$f(x) = \sin(\sqrt{2^x}).$$

9. Compute $\frac{dy}{dx}$ given that

$$\sin(xy) = x.$$

Find an equation for the line that is tangent to this curve at the point $(\frac{1}{\sqrt{2}}, \frac{\sqrt{2}\pi}{4})$.

10. Find $\frac{d^2y}{dx^2}$ if

$$xy - e^y = 0.$$

11. The volumetric flow rate Q (the volume of fluid passing through a given surface per unit time) of blood moving through a cylindrical blood vessel with radius r is given by the Hagen-Poiseuille equation

$$Q = \frac{\pi r^4}{8\mu L}\Delta p,$$

where μ denotes blood viscosity, L denotes the length of the vessel, and Δp denotes the change in pressure along the vessel. Assume μ, L, and Δp are held constant, and that the radius is decreasing at a constant rate $-.1$ mm/s. Find the rate at which Q is decreasing when $r = 2$ mm.

12. In Example 6 of Section 4.8, the author of our textbook discusses an allometric relationship between the area A of a leaf and its stem diameter D:

$$A = cD^{1.84},$$

where c is a constant of proportionality. Find a relationship between the relative growth rate of the leaf area (i.e., $\frac{A'(t)}{A(t)}$) and the relative growth rate of the stem diameter (i.e., $\frac{D'(t)}{D(t)}$).

13. An airplane is flying 6 miles above the ground on a flight path that will take it directly over a radar tracking station. If the distance between the plane and tracking station is decreasing at a rate of 400 miles per hour when the distance is 10 miles, what is the velocity of the plane?

14. Let

$$f(x) = \cos x - \sin x, \quad -\frac{\pi}{4} \leq x \leq \frac{3\pi}{4},$$

and compute $\frac{df^{-1}}{dx}(1)$.

15. Evaluate the expression

$$\tan(\sin^{-1}(\frac{2}{3})).$$

16. Show that

$$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}, \quad -1 < x < +1.$$

17. Let

$$y = x^{\tan x}, \quad 0 \le x < \frac{\pi}{2},$$

and compute $\frac{dy}{dx}$.

18. Find the linearization of $f(x) = \sqrt[3]{x}$ at $a = 8$, and use it to approximate $\sqrt[3]{9}$.

19. Use a linear approximation to estimate a value for

$$\ln(.99).$$

20. Suppose a measurement $x = 3 \pm .1$ is made, and we compute $f(3) = \frac{1}{4}$, where

$$f(x) = \frac{1}{(x-1)^2}.$$

Approximate the absolute error on f $|\Delta f|$, the relative error $|\frac{\Delta f}{f(3)}|$, and the percentage error $|\frac{\Delta f}{f(3)}|100$.

21. Consider a right triangle with hypotenuse length l and sidelengths 3 and x. Suppose x is measured as $x = 4 \pm .05$, and use linear approximation to approximate the associated range of error on l. More precisely, find $|\Delta l|$ and give the interval $[l(4) - |\Delta l|, l(4) + |\Delta l|]$.

Solutions

1. According to the definition,

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 \cos \frac{1}{h}}{h} = \lim_{h \to 0} h \cos \frac{1}{h}.$$

Observing now that $|\cos \frac{1}{h}| \le 1$, we have

$$-|h| \le h \cos \frac{1}{h} \le |h|,$$

and so by the squeeze theorem $f'(0) = 0$. (This is a case in which $f(x)$ is differentiable at a point, but $f'(x)$ is not continuous at that point.)

2a. We can see that $f(x)$ is not continuous at $x = 0$ by computing

$$\lim_{x \to 0^-} f(x) = 1,$$

and

$$\lim_{x \to 0^+} f(x) = -1,$$

from which we conclude that the limit of $f(x)$ as $x \to 0$ does not exist. We know that if a function is not continuous at a point then it cannot be differentiable at that point, so $f(x)$ is not differentiable at $x = 0$.

2b. **Method 1.** According to the definition of derivative,

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h}.$$

Since f is defined by different functions for $h < 0$ and for $h > 0$, we must compute and compare right and left limits:

$$\lim_{h\to 0^-} \frac{(h^2+1) - 1}{h} = \lim_{h\to 0^-} h = 0,$$

while

$$\lim_{h\to 0^+} \frac{(2h+1) - 1}{h} = \lim_{h\to 0^+} 2 = 2.$$

Since these limits do not agree, we can conclude that $f(x)$ is not differentiable at $x = 0$.

Method 2. In cases for which

$$f(x) = \begin{cases} f_1(x) & x \le a \\ f_2(x) & x > a \end{cases},$$

where $f_1(a) = f_2(a)$, $f_1'(a) = c_1$, and $f_2'(a) = c_2$ we can proceed as follows: if $c_1 \neq c_2$ then f is not differentiable at $x = a$, while if $c_1 = c_2$ then f is differentiable at $x = a$ and $f'(a) = c_1$. Here, $f_1(x) = x^2 + 1$ and $f_2(x) = 2x + 1$, so $f'(0) = 0$ while $f_2'(0) = 2$. We can draw the same conclusion as we did with Method 1. (Be sure to check all assumptions when using this method; try it, for example, on (2a).)

2c. In this case,

$$f'(0) = \lim_{h\to 0} \frac{h|h|}{h} = \lim_{h\to 0} |h| = 0,$$

and so f is differentiable at $x = 0$ with $f'(0) = 0$.

3. The slope of the tangent line is given by the derivative $y'(1) = 3(1)^2 = 3$. We have, then $y - 2 = 3(x - 1)$. See Figure 4.

4a. The graphs are sketched below.

4b. The average velocity is

$$v_{avg} = \frac{s(2) - s(0)}{2 - 0} = \frac{80 - 0}{2} = 40\frac{\text{km}}{\text{hr}}.$$

4c. The instantaneous velocity is

$$v_{inst} = s'(1) = 40\frac{\text{km}}{\text{hr}}.$$

5. The slope of the tangent line to this curve at any point x is

$$f'(x) = 2x + 1,$$

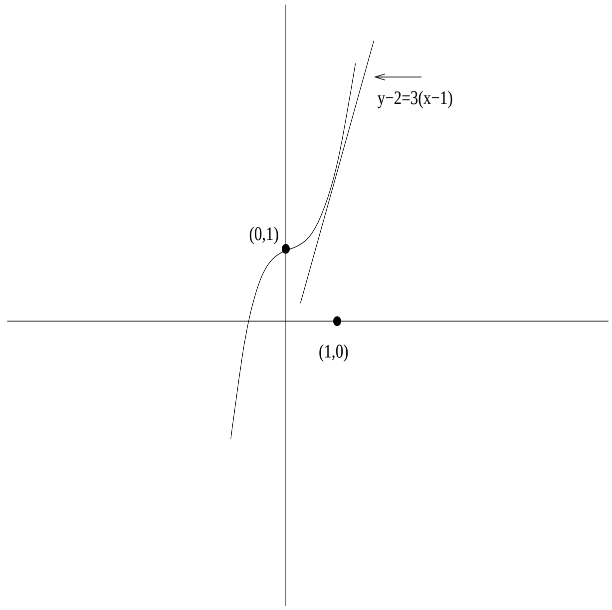

Figure 1: Figure for Problem 3.

and we must find the value x for which this slope is 3 (the slope of the given line). We solve

$$2x + 1 = 3$$

to find $x = 1$. The point is $(1, 3)$.

6.

6a. Applying the power rule to each summand, we find

$$\frac{d}{dx}(x^{\frac{2}{3}} + x^{-7}) = \frac{2}{3}x^{-\frac{1}{3}} - 7x^{-8}.$$

6b. Applying the product rule, we find

$$\frac{d}{dx}x \sin x = \sin x + x \cos x.$$

6c. Applying the quotient rule, we find

$$\frac{d}{dx}\frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2} = \frac{4}{(e^x + e^{-x})^2},$$

where the final expression was obtained by multiplying out terms in the numerator.

6d. Proceeding with the chain rule, we compute

$$\frac{d}{dx}(2x + \frac{1}{x})^2 = 2(2x + \frac{1}{x})(2 - \frac{1}{x^2}) = 8x - \frac{2}{x^3}.$$

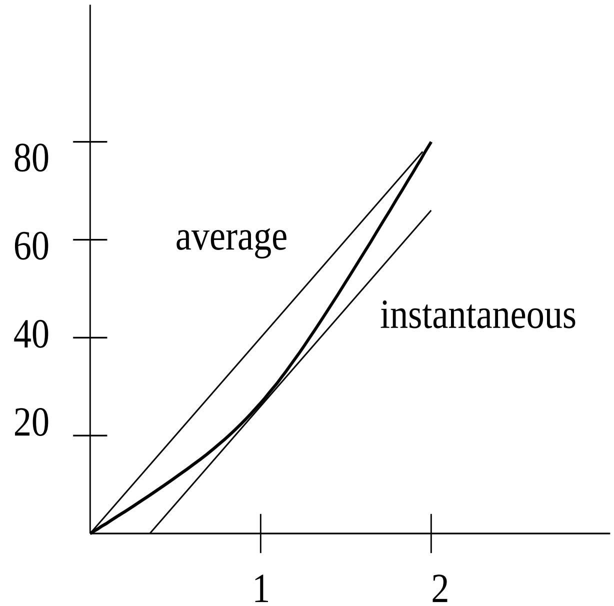

Figure 2: Figure for Problem 4.

6e. Proceeding with the chain rule, we compute

$$\frac{d}{dx}\sin^{-1}\frac{5}{x^2} = \frac{1}{\sqrt{1-(\frac{5}{x^2})^2}}(-\frac{10}{x^3}) = -\frac{10}{x^3\sqrt{1-\frac{25}{x^4}}}.$$

6f. proceeding with the chain rule, we compute

$$\frac{d}{dx}\tan^{-1}\sqrt{1-x^4} = \frac{1}{1-x^4+1}\cdot\frac{1}{2\sqrt{1-x^4}}(-4x^3) = -\frac{2x^3}{(2-x^4)\sqrt{1-x^4}}.$$

7. First,

$$h'(x) = f'(x)e^{g(x)} + f(x)e^{g(x)}g'(x),$$

and so

$$h'(2) = f'(2)e^{g(2)} + f(2)e^{g(2)}g'(2) = 7e^0 + 4e^0 3 = 7 + 12 = 19.$$

8. **Method 1.** Compute directly

$$f'(x) = \cos(\sqrt{2^x})\frac{1}{2\sqrt{2^x}}2^x\ln 2 = \frac{\ln 2}{2}\cos(\sqrt{2^x})\sqrt{2^x},$$

and

$$\begin{aligned} f''(x) &= \frac{\ln 2}{2}\Big(-\sin(\sqrt{2^x})\frac{\ln 2}{2}2^x + \cos(\sqrt{2^x})\frac{\ln 2}{2}\sqrt{2^x}\Big) \\ &= (\frac{\ln 2}{2})^2\Big(\sqrt{2^x}\cos(\sqrt{2^x}) - 2^x\sin(\sqrt{2^x})\Big). \end{aligned}$$

Method 2. First, observe that $\sqrt{2^x} = (\sqrt{2})^x$, which eliminates the need for a nested chain rule. Now,

$$\frac{d}{dx}\sin((\sqrt{2})^x) = \cos((\sqrt{2})^x)(\sqrt{2})^x \ln\sqrt{2},$$

and

$$\begin{aligned}\frac{d^2}{dx^2}\sin((\sqrt{2})^x) &= -\sin((\sqrt{2})^x)((\sqrt{2})^x \ln\sqrt{2})^2 + \cos((\sqrt{2})^x)((\sqrt{2})^x \ln\sqrt{2})\ln\sqrt{2} \\ &= (\ln\sqrt{2})^2\Big(\cos((\sqrt{2})^x)(\sqrt{2})^x - \sin((\sqrt{2})^x)2^x\Big),\end{aligned}$$

which is equivalent to the expression from Method 1.

9. We compute implicitly

$$\frac{d}{dx}\sin(xy) = \frac{d}{dx}x \Rightarrow \cos(xy)\frac{d}{dx}(xy) = 1 \Rightarrow \cos(xy)(y + x\frac{dy}{dx}) = 1.$$

Solving for $\frac{dy}{dx}$, we find

$$\frac{dy}{dx} = \frac{\frac{1}{\cos(xy)} - y}{x}.$$

At the point $(\frac{1}{\sqrt{2}}, \frac{\sqrt{2}\pi}{4})$, we have

$$\frac{dy}{dx} = \frac{\frac{1}{\cos(\frac{\pi}{4})} - \frac{\sqrt{2}\pi}{4}}{\frac{1}{\sqrt{2}}} = 2 - \frac{\pi}{2}.$$

The equation for the tangent line is

$$(y - \frac{\sqrt{2}\pi}{4}) = (2 - \frac{\pi}{2})(x - \frac{1}{\sqrt{2}}).$$

10. We begin by computing the x-derivative of the entire equation,

$$y + x\frac{dy}{dx} - e^y\frac{dy}{dx} = 0.$$

Solving for $\frac{dy}{dx}$, we obtain

$$\frac{dy}{dx} = -\frac{y}{x - e^y}.$$

We now compute the second derivative directly from this expression:

$$\begin{aligned}\frac{d^2y}{dx^2} &= -\frac{\frac{dy}{dx}(x - e^y) - y(1 - e^y\frac{dy}{dx})}{(x - e^y)^2} = -\frac{(-\frac{y}{x-e^y})(x - e^y) - y + ye^y(-\frac{y}{x-e^y})}{(x - e^y)^2} \\ &= \frac{2y(x - e^y) + y^2e^y}{(x - e^y)^3}.\end{aligned}$$

11. First, we compute a derivative with respect to t on both sides, which gives

$$\frac{dQ}{dt} = \frac{\pi \Delta p}{8\mu L} \cdot 4r^3 \frac{dr}{dt} = \frac{\pi \Delta p}{2\mu L} r^3 \frac{dr}{dt}.$$

Now we substitute in $r = 2$ mm and $\frac{dr}{dt} = -.1$ mm/s to get

$$\frac{dQ}{dt} = \frac{\pi \Delta p}{2\mu L} 2^3(-.1) = \frac{4\pi \Delta p}{\mu L}(-.1) = -\frac{.4\pi \Delta p}{\mu L} \text{ mm}^3/\text{s}.$$

12. We begin by computing

$$A'(t) = c1.84D^{1.84-1}D'(t) = 1.84(cD^{1.84})\frac{D'(t)}{D(t)}.$$

Here, we observe that $A(t) = cD^{1.84}$, so that we have

$$A'(t) = 1.84A(t)\frac{D'(t)}{D(t)} \Rightarrow \frac{A'(t)}{A(t)} = 1.84\frac{D'(t)}{D(t)}.$$

We see that the relative growth rate for the area is 1.84 times the relative growth rate of the stem diameter.

13. In this case, we are given that $\frac{dz}{dt} = -400$, where z denotes the distance between the plane and the tracking station. If we let x denote the horizontal distance between the plane and the tracking station, then what we are looking for is $\frac{dx}{dt}$, the plane's velocity. (See Figure 3.)

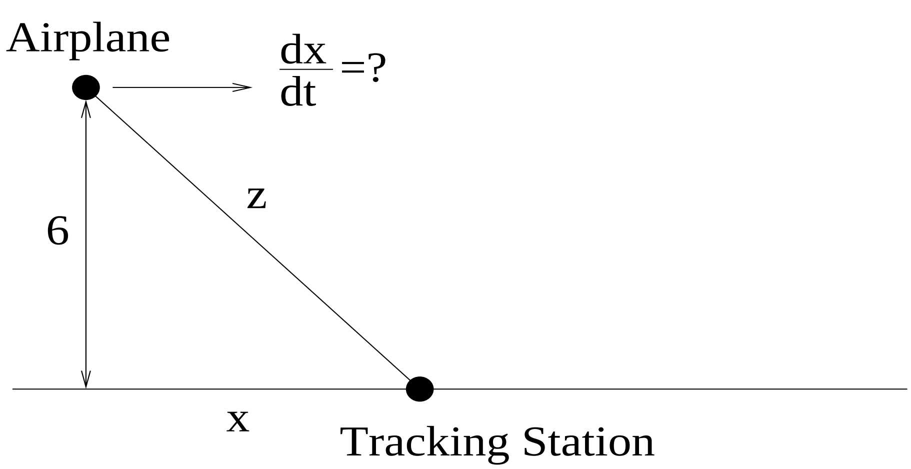

Figure 3: Figure for Problem 13.

In order to find a relation between $\frac{dx}{dt}$ and $\frac{dz}{dt}$, we begin by relating x and z. We have

$$x^2 + 36 = z^2.$$

Upon differentiation of this equation with respect to t, we find

$$2x\frac{dx}{dt} = 2z\frac{dz}{dt}.$$

When $z = 10$, we have $x = \sqrt{100 - 36} = 8$, and therefore

$$2(8)\frac{dx}{dt} = 2(10)(-400) \Rightarrow \frac{dx}{dt} = -\frac{8000}{16} = -500 \text{ mph}.$$

The negative sign indicates that the plane is moving toward the tracking station.

14. First,

$$f'(x) = -\sin x - \cos x.$$

Also, $f(0) = 1 \Rightarrow f^{-1}(1) = 0$. We have, then,

$$\frac{df^{-1}}{dx}(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{-1} = -1.$$

15. For calculations like this it's often convenient to set

$$\theta = \sin^{-1}\frac{2}{3}$$

(using θ because this is an angle), so that

$$\sin\theta = \frac{2}{3}.$$

(See Figure 4.)

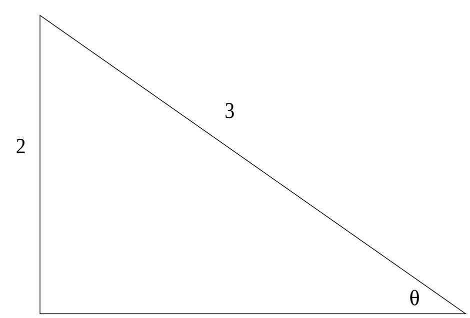

Figure 4: Figure for Problem 15.

According to the Pythagorean Theorem, the adjacent sidelength is

$$b = \sqrt{9 - 4} = \sqrt{5}.$$

In this way,

$$\tan(\sin^{-1}\frac{2}{3}) = \tan\theta = \frac{2}{\sqrt{5}}.$$

16. Set $f(x) = \cos x$ and use the formula

$$\frac{df^{-1}}{dx}(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{-\sin(\cos^{-1}x)}.$$

In order to evaluate $\cos^{-1} x$, set $\theta = \cos^{-1} x$ (we use θ because this is an angle) and note that consequently

$$\cos\theta = x \Rightarrow \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-x^2}.$$

Notice here that since the range of $\cos^{-1} x$ is $[0,\pi]$, we know that $\theta \in [0,\pi]$, and so we know $\sin\theta \geq 0$. This chooses the sign in front of $\sqrt{1-\cos^2\theta}$. We finally have

$$\frac{1}{-\sin(\cos^{-1} x)} = -\frac{1}{\sqrt{1-x^2}}.$$

17. If we take the natural logarithm of both sides, we have

$$\ln y = \ln x^{\tan x} = (\tan x)(\ln x).$$

Now differentiate each side with respect to x to obtain

$$\frac{1}{y}\frac{dy}{dx} = (\sec^2 x)(\ln x) + \frac{\tan x}{x}.$$

Multiplying this last expression by $y = x^{\tan x}$, we conclude

$$\frac{dy}{dx} = x^{\tan x}((\sec^2 x)(\ln x) + \frac{\tan x}{x}).$$

18. First, the linearization of $f(x)$ at any value a is

$$L(x) = f(a) + f'(a)(x-a).$$

In this case $a = 8$, and so

$$L(x) = f(8) + f'(8)(x-8).$$

Here, $f(8) = \sqrt[3]{8} = 2$, and

$$f'(x) = \frac{1}{3}x^{-2/3},$$

so that $f'(8) = \frac{1}{3}\frac{1}{4} = \frac{1}{12}$. The linearization is

$$L(x) = 2 + \frac{1}{12}(x-8).$$

Finally, we approximate $\sqrt[3]{9}$ with

$$L(9) = 2 + \frac{1}{12} = \frac{25}{12}.$$

19. We start with $f(x) = \ln x$ and use the linear approximation

$$L(x) = f(a) + f'(a)(x-a),$$

where it is reasonable here to take $a = 1$ (because $f(1)$ is easy to compute). We find

$$L(x) = \ln 1 + 1(x-1) = x - 1.$$

We can now compute

$$f(.99) \approx .99 - 1 = -.01.$$

20. The error formula is

$$|\Delta f| \approx |f'(3)||\Delta x|,$$

and the error is given as $|\Delta x| \leq .1$. In this case,

$$f'(x) = -\frac{2}{(x-1)^3},$$

so that $f'(3) = -\frac{2}{8} = -\frac{1}{4}$. This gives

$$|\Delta f| \approx \frac{1}{4}|\Delta x| \leq \frac{.1}{4} = \frac{1}{40} = .025.$$

The relative error is

$$|\frac{\Delta f}{f(3)}| \leq \frac{\frac{1}{40}}{\frac{1}{4}} = \frac{1}{10} = .1,$$

and the percentage error is 10%.

21. First, the length l is given by the Pythagorean Theorem,

$$l = \sqrt{3^2 + x^2}.$$

The error formula for l is

$$|\Delta l| \approx |l'(4)||\Delta x|,$$

where $|\Delta x| \leq .05$. Here,

$$l'(x) = \frac{x}{\sqrt{x^2+9}},$$

so $l'(4) = \frac{4}{\sqrt{25}} = \frac{4}{5}$, and

$$|l'(4)||\Delta x| \leq \frac{4}{5}(.05) = .04.$$

The interval is

$$[5 - .04, 5 + .04] = [4.96, 5.04].$$