M151B, Fall 2008, Practice Problems for Exam 2

Calculators will not be allowed on the exam.

1. Let

$$f(x) = \cos x - \sin x, \quad -\frac{\pi}{4} \leq x \leq \frac{3\pi}{4},$$

and compute $\frac{df^{-1}}{dx}(1)$.

2. Show that

$$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}, \quad -1 < x < +1.$$

3. Let

$$y = x^{\tan x}, \quad 0 \leq x < \frac{\pi}{2},$$

and compute $\frac{dy}{dx}$.

4. Use a linear approximation to estimate a value for

$$\ln(.99).$$

5. Consider a right triangle with hypotenuse length l and sidelengths 3 and x. Suppose x is measured as $x = 4 \pm .05$, and use linear approximation to approximate the associated range of error on l.

6. Suppose that $f(x)$ is continuous on the interval $[2, 5]$ and differentiable on the interval $(2, 5)$. Show that if $1 \leq f'(x) \leq 4$ for all $x \in [2, 5]$, then $3 \leq f(5) - f(2) \leq 12$.

7. Suppose that $f(x)$ is twice differentiable in an open interval containing the point $x = c$ and has a local minimum at the same point. Show that the function $g(x) = e^{f(x)}$ has a local minimum at $x = c$.

8. Let

$$f(x) = \frac{e^{-x}}{x-1}, \quad x \neq 1.$$

8a. Locate the critical points of f and determine the intervals on which f is increasing and the intervals on which f is decreasing.

8b. Locate the possible inflection points for f and determine the intervals on which f is concave up and the intervals on which it is concave down.

8c. Determine the boundary behavior of f by computing limits as $x \to \pm\infty$.

8d. Use your information from Parts a-c to sketch a graph of this function.

9. A long rectangular sheet of metal, 12 inches wide, is to be made into a rain gutter by turning up two sides at right angles to the sheet. How many inches should be turned up to give the gutter its greatest capacity?

10. A piece of wire 10 meters long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is minimized? How should the wire be cut so that the total area is maximized?

11. Compute the following limits.

11a.

$$\lim_{x \to 0} \frac{e^x - 1}{x}.$$

11b.

$$\lim_{x \to \infty} (x - \sqrt{x^2 - 1}).$$

11c.

$$\lim_{x \to \infty} (\frac{x}{x+1})^x.$$

Solutions

1. First,

$$f'(x) = -\sin x - \cos x.$$

Also, $f(0) = 1 \Rightarrow f^{-1}(1) = 0$. We have, then,

$$\frac{df^{-1}}{dx}(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{-1} = -1.$$

2. Set $f(x) = \cos x$ and use the formula

$$\frac{df^{-1}}{dx}(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{-\sin(\cos^{-1} x)}.$$

In order to evaluate $\cos^{-1} x$, set $\theta = \cos^{-1} x$ (we use θ because this is an angle) and note that consequently

$$\cos\theta = x \Rightarrow \sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - x^2}.$$

Notice here that since the range of $\cos^{-1} x$ is $[0, \pi]$, we know that $\theta \in [0, \pi]$, and so we know $\sin\theta \geq 0$. This chooses the sign in front of $\sqrt{1 - \cos^2\theta}$. We finally have

$$\frac{1}{-\sin(\cos^{-1} x)} = -\frac{1}{\sqrt{1 - x^2}}.$$

3. If we take the natural logarithm of both sides, we have

$$\ln y = \ln x^{\tan x} = (\tan x)(\ln x).$$

Now differentiate each side with respect to x to obtain

$$\frac{1}{y}\frac{dy}{dx} = (\sec^2 x)(\ln x) + \frac{\tan x}{x}.$$

Multiplying this last expression by $y = x^{\tan x}$, we conclude

$$\frac{dy}{dx} = x^{\tan x}((\sec^2 x)(\ln x) + \frac{\tan x}{x}).$$

4. We start with $f(x) = \ln x$ and use the linear approximation

$$f(x) = f(a) + f'(x)(x - a),$$

where it is reasonable here to take $a = 1$. We find

$$f(x) \approx \ln 1 + 1(x - 1) = x - 1.$$

We can now compute

$$f(.99) \approx .99 - 1 = -.01.$$

(The exact value, to four decimal places, is -.0101.)

5. First, the length l is given by the Pythagorean Theorem,

$$l = \sqrt{3^2 + x^2}.$$

By linear approximation, we have

$$l(x + \Delta x) - l(x) \approx l'(x)\Delta x,$$

where $|l(x + \Delta x) - l(x)|$ is the absolute error, $x = 4$, $\Delta x = .05$ and

$$l'(x) = \frac{x}{\sqrt{x^2 + 9}}.$$

We have, then,

$$l'(4)(.05) = \frac{4}{5}(.05) = .04.$$

We conclude

$$l(4 \pm .05) = 5 \pm .04.$$

6. By the Mean Value Theorem, we know that there exists some value $c \in (2, 5)$ so that

$$f'(c) = \frac{f(5) - f(2)}{3}.$$

Since the largest possible value for $f'(c)$ on this interval is 4 and since the smallest possible value for $f'(c)$ on this interval is 1, we have the inequality

$$1 \leq \frac{f(5) - f(2)}{3} \leq 4.$$

Multiplying this last inequality by 3, we find

$$3 \leq f(5) - f(2) \leq 12.$$

7. By our assumptions on f, we know $f'(c) = 0$ and $f''(c) > 0$. We need to show that precisely the same two conditions hold for $g(x) = e^{f(x)}$. We have, first

$$g'(c) = e^{f(c)} f'(c) = 0,$$

and second

$$g''(c) = e^{f(c)} f'(c)^2 + e^{f(c)} f''(c) = e^{f(c)} f''(c) > 0,$$

where in arriving at this last inequality we have observed that $e^{f(x)} > 0$ for any finite value of $f(x)$.

8a. Compute

$$f'(x) = -\frac{xe^{-x}}{(x-1)^2},$$

and observe that the critical points are $x = 0, 1$. We see that f is increasing on $(-\infty, 0]$ and decreasing on $[0, 1) \cup (1, \infty)$. (We omit $x = 1$ because f isn't defined there.)

8b. Compute

$$f''(x) = \frac{e^{-x}(x^2+1)}{(x-1)^3},$$

and observe that the only possible inflection point is $x = 1$. We see that $f(x)$ is concave down on $(-\infty, 1)$ and concave up on $(1, \infty)$.

8c. For the boundary behavior, we have

$$\begin{aligned} \lim_{x \to -\infty} \frac{e^{-x}}{x-1} &= -\infty \\ \lim_{x \to +\infty} \frac{e^{-x}}{x-1} &= 0. \end{aligned}$$

8d. In order to anchor the plot, we evaluate f at the critical points, the possible inflection points and the endpoints. We have

$$\begin{aligned} f(0) &= -1 \\ \lim_{x \to 1^-} \frac{e^{-x}}{x-1} &= -\infty \\ \lim_{x \to 1^+} \frac{e^{-x}}{x-1} &= +\infty. \end{aligned}$$

The boundary behavior was obtained in Part c. Here's a sketch:

9. Let x denote the width of sheet to be turned up on one side and let y denote the width of sheet left flat. If the length of the sheet is L then the volume is

$$V = xyL,$$

where y can be eliminated by the relation $y = 12 - 2x$. (For this problem it seems fairly natural to avoid bringing up the variable y, but I've used it here for consistency with our standard process.) In this way the function we would like to maximize is

$$V(x) = x(12 - 2x)L, \quad 0 \le x \le 6.$$

We find the critical points by computing

$$\frac{dV}{dx} = (12 - 4x)L = 0 \Rightarrow x = 3.$$

Evaluating

$$\begin{aligned} V(0) &= 0 \\ V(3) &= 18 \\ V(6) &= 0, \end{aligned}$$

we conclude that the maximum capacity occurs when $x = 3$ inches are turned up on either side.

10. Let x be the length of each side of the square, and let y be the length of each side of the equilateral triangle. Then the total length of wire is

$$10 = 4x + 3y,$$

while the total area is

$$A = \text{area of square } + \text{ area of triangle} = x^2 + \frac{\sqrt{3}}{4}y^2.$$

(You can derive the area formula for an equilateral triangle from the formula $\frac{1}{2}bh$ and either the sidelengths for a 30-60-90 triangle or the Pythagorean theorem.) Solving our constraint for y, we have

$$y = \frac{10}{3} - \frac{4}{3}x,$$

so that

$$A(x) = x^2 + \frac{\sqrt{3}}{4}(\frac{10}{3} - \frac{4}{3}x)^2, \quad 0 \leq x \leq \frac{10}{4}.$$

Proceeding as usual, we compute

$$A'(x) = 2x + \frac{\sqrt{3}}{2}(\frac{10}{3} - \frac{4}{3}x)(-\frac{4}{3}) \Rightarrow x(2 + \frac{8\sqrt{3}}{9}) = \frac{20\sqrt{3}}{9}.$$

We conclude that

$$x = \frac{20\sqrt{3}}{18 + 8\sqrt{3}} = \frac{10\sqrt{3}}{9 + 4\sqrt{3}}.$$

From our expression for $A'(x)$ we see that $A'(x) < 0$ for $x < \frac{10\sqrt{3}}{9+4\sqrt{3}}$, while $A'(x) > 0$ for $x < \frac{10\sqrt{3}}{9+4\sqrt{3}}$. We conclude that $A(x)$ decreases for all x to the left of this point and increases for all x to the right of it, and is consequently a global minimum. This says that the area is minimized if the length of wire taken for the square is

$$4x = \frac{40\sqrt{3}}{9 + 4\sqrt{3}}.$$

In order to find the global maximum, we must check $A(x)$ at the two endpoints. We have

$$A(0) = \frac{100\sqrt{3}}{36}$$
$$A(\frac{10}{4}) = \frac{100}{16}.$$

Clearly, $A(\frac{10}{4})$ is larger, and this corresponds with putting all of the wire into the square.

11a.

$$\lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{x\to 0} \frac{e^x}{1} = 1.$$

11b.

$$\lim_{x\to\infty} (x - \sqrt{x^2 - 1}) = \lim_{x\to\infty} x(1 - \sqrt{1 - \frac{1}{x^2}}) = \lim_{x\to\infty} \frac{1 - \sqrt{1 - \frac{1}{x^2}}}{\frac{1}{x}}.$$

We can now apply l'Hospital's rule to find that this limit is

$$\lim_{x\to\infty} \frac{-\frac{1}{2}(1 - \frac{1}{x^2})^{-\frac{1}{2}}(\frac{2}{x^3})}{-\frac{1}{x^2}} = 0.$$

11c.

$$\lim_{x\to\infty} (\frac{x}{x+1})^x = \lim_{x\to\infty} e^{\ln(\frac{x}{x+1})^x} = \lim_{x\to\infty} e^{x\ln(\frac{x}{x+1})} = e^{\lim_{x\to\infty} x\ln(\frac{x}{x+1})}.$$

We compute the limit in the exponent as

$$\lim_{x\to\infty} \frac{\ln(\frac{x}{x+1})}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\ln x - \ln(x+1)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{1}{x} - \frac{1}{x+1}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{\frac{1}{x(x+1)}}{-\frac{1}{x^2}} = \lim_{x\to\infty} -\frac{x^2}{x^2+x} = -1.$$

We conclude

$$\lim_{x\to\infty} (\frac{x}{x+1})^x = e^{-1}.$$

Note: It's slicker—but for our purposes less instructive—to simply notice that this is the inverse of the limit

$$e = \lim_{x\to\infty} (1 + \frac{1}{x})^x.$$