M611 Fall 2014 Assignment 11, due Friday Dec. 5

1. [10 pts] (**Evans 2.5.19.**)

a. Show the general solution of the PDE $u_{xy} = 0$ is

$$u(x, y) = F(x) + G(y)$$

for arbitrary functions F, G.

b. Using the change of variables $\xi = x + t$, $\eta = x - t$, show $u_{tt} - u_{xx} = 0$ if and only if $u_{\xi\eta} = 0$.

c. Use (a) and (b) to rederive d'Alembert's formula.

d. Under what conditions on the initial data g, h is the solution u a right-moving wave? A left-moving wave?

2. [10 pts] Solve the following:

a. Solve the quarter-plane problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx}; \quad (x,t) \in (0,\infty) \times (0,\infty) \\ u_x(0,t) &= 0; \quad t \geq 0 \\ u(x,0) &= g(x); \quad x \geq 0 \\ u_t(x,0) &= h(x); \quad x \geq 0. \end{aligned}$$

Notice that the difference between this problem and the quarter-plane problem we solved in class is the condition $u_x(0,t) = 0$ (replacing $u(0,t) = 0$).

b. Solve the equation from Part (a) with $c = 2$, $h(x) = 0$, and

$$g(x) = \begin{cases} x - 2 & 2 \leq x \leq 3 \\ 4 - x & 3 < x \leq 4 \\ 0 & \text{otherwise} \end{cases}.$$

Sketch graphs of $u(x,0)$, $u(x,1)$, and $u(x,2)$.

c. Solve the equation from Part (a) with $c = 2$, $g(x) = 0$ and

$$h(x) = \frac{1}{x^2 + 1}.$$

Sketch a graph of $u(x,1)$.

Note. For Part (a) you can proceed similarly as in Problem 1, beginning with

$$u(x,t) = F(x - ct) + G(x + ct).$$

3. [10 pts] (**Evans 2.5.21.**)

a. Assume $\vec{E} = (E^1, E^2, E^3)$ and $\vec{B} = (B^1, B^2, B^3)$ solve Maxwell's equations

$$\begin{aligned} \vec{E}_t &= \nabla \times \vec{B} \\ \vec{B}_t &= -\nabla \times \vec{E} \\ \nabla \cdot \vec{E} &= 0 \\ \nabla \cdot \vec{B} &= 0. \end{aligned}$$

Show

$$\vec{E}_{tt} - \Delta \vec{E} = 0, \quad \vec{B}_{tt} - \Delta \vec{B} = 0.$$

b. Assume that $\vec{u} = (u^1, u^2, u^3)$ solves the evolution equations of linear elasticity

$$\vec{u}_{tt} - \mu \Delta \vec{u} - (\lambda + \mu) D(\text{div}\vec{u}) = 0 \quad \text{in } \mathbb{R}^3 \times (0, \infty).$$

Show $w := \text{div}\ \vec{u}$ and $\vec{w} := \text{curl}\ \vec{u}$ each solve wave equations, but with different speeds of propagation.

4. [10 pts] As in the statement of Lemma 2.4.1 assume $u \in C^m(\mathbb{R}^n \times [0, \infty))$, $m, n \geq 2$, and

$$U(r, t; \vec{x}) = \fint_{\partial B(\vec{x}, r)} u(\vec{y}, t) dS_y.$$

(In fact, this problem only requires $m = 2$.) Verify the following relations, claimed in the proof of Lemma 2.4.1:

$$\begin{aligned} U_r(r, t; \vec{x}) &= \frac{r}{n} \fint_{B(\vec{x}, r)} \Delta_y u(\vec{y}, t) d\vec{y} \\ U_{rr}(r, t; \vec{x}) &= \frac{1 - n}{n} \fint_{B(\vec{x}, r)} \Delta_y u(\vec{y}, t) d\vec{y} + \fint_{\partial B(\vec{x}, r)} \Delta_y u(\vec{y}, t) dS_y. \end{aligned}$$