M611 Fall 2019, Assignment 4, due Friday Sept. 27

1. [10 pts] Show that the Lebesgue dominated convergence theorem fails to apply for the sequence of functions $f_n(x) = n^2 x e^{-nx}$ for $x \in [0, 1]$.

2. [10 pts] Prove the following theorem: Suppose $U \subset \mathbb{R}^n$ is open and $1 < p, q < \infty$, with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} < 1$. Show that if $f \in L^p(U)$ and $g \in L^q(U)$ then

$$\|fg\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

3. [10 pts] Show that if $f \in L^1(\mathbb{R}^n)$ then given any $\epsilon > 0$ there exists $\delta > 0$ so that

$$\Big|E\Big| < \delta \Rightarrow \int_E |f| d\vec{x} < \epsilon.$$

4. [10 pts] Proceed as in our Proof of Theorem A.C.7 (i) (on mollifiers) to prove the following lemma, which will be useful in our analysis of Poisson's equation.

Lemma. Suppose $\phi \in C_c^2(\mathbb{R}^n)$ and $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and set $u(\vec{x}) = \phi * f(\vec{x})$. Prove that for each $i, j = 1, 2, \ldots, n$

$$u_{x_i x_j}(\vec{x}) = \phi_{x_i x_j} * f(\vec{x}),$$

for all $\vec{x} \in \mathbb{R}^n$ and that $u_{x_i x_j}$ is a continuous function on $\mathbb{R}^n$ (i.e., $u \in C^2(\mathbb{R}^n)$).