M611 Fall 2019 Assignment 5, due Friday Oct. 4

1. [10 pts] We define the convolution of two functions $f, g : \mathbb{R}^n \to \mathbb{R}$ by

$$f * g(\vec{x}) := \int_{\mathbb{R}^n} f(\vec{x} - \vec{y}) g(\vec{y}) d\vec{y}.$$

Establish the following properties of convolution:
(i) For $f, g \in C_c^\infty(\mathbb{R}^n)$

$$f * g = g * f$$

(ii)

$$f, g \in L^2(\mathbb{R}^n) \Rightarrow \|f * g\|_{L^\infty} \leq \|f\|_{L^2} \|g\|_{L^2}$$

(iii)

$$f, g \in L^1(\mathbb{R}^n) \Rightarrow \|f * g\|_{L^1} \leq \|f\|_{L^1} \|g\|_{L^1}$$

(iv)

$$f, g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \Rightarrow f * g \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n).$$

Note. In proving Items (ii) and (iii), you will see that if either $f, g \in L^2(\mathbb{R}^n)$ (Item (ii)) or $f, g \in L^1(\mathbb{R}^n)$ (Item (iii)), then $f(\vec{x} - \cdot) g(\cdot) \in L^1(\mathbb{R}^n)$ for a.e. $\vec{x} \in \mathbb{R}^n$. It follows immediately that the symmetry stated in Item (i) holds in those cases as well.

2. [10 pts] (**Evans 1.5.3.**) Prove the Multinomial Theorem

$$(x_1 + x_2 + \cdots + x_n)^k = \sum_{|\alpha| = k} \binom{|\alpha|}{\alpha} \vec{x}^\alpha,$$

where $\binom{|\alpha|}{\alpha} := \frac{|\alpha|!}{\alpha!}$, $\alpha! = \alpha_1! \alpha_2! \dots \alpha_n!$, and $\vec{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. The sum is taken over all multiindices $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ with $|\alpha| = k$.

3. [10 pts] Let $B^o(0, r)$ denote the open ball of radius r in $\mathbb{R}^n$, and take as definitions

$$|B^o(0, r)| := \int_{B^o(0,r)} 1 d\vec{x}$$
$$|\partial B^o(0, r)| := \int_{\partial B^o(0,r)} 1 dS.$$

a. Show that

$$|B^o(0, r)| = |B^o(0, 1)| r^n,$$

and likewise

$$|\partial B^o(0, r)| = |\partial B^o(0, 1)| r^{n-1}.$$

b. Show that

$$|\partial B^o(0, r)| = \frac{n}{r} |B^o(0, r)|.$$

c. Show that

$$|\partial B^o(0, 1)| = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})},$$

where Γ denotes the gamma function

$$\Gamma(k) = \int_0^\infty t^{k-1} e^{-t} dt.$$

(Hint. Use the integral equality $\int_{\mathbb{R}^n} e^{-|\vec{x}|^2} d\vec{x} = \pi^{n/2}$, and evaluate the integral in polar coordinates.)

d. Show that

$$\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} = \frac{n\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)},$$

so that we find the volumes:

$$|\partial B^o(0,1)| = n\alpha(n),$$

where

$$\alpha(n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}.$$

Note that $|B^o(0,1)| = \alpha(n)$ and this could just as well have been stated for the closed ball $B(0,r)$.

4. [10 pts] (**Evans 1.5.5.**) Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is smooth. Prove

$$f(\vec{x}) = \sum_{|\alpha| \leq k} \frac{D^\alpha f(0)}{\alpha!} \vec{x}^\alpha + \mathbf{O}(|\vec{x}|^{k+1}).$$

Note. In fact, let's be a bit more precise: Suppose $f \in C^{k+1}(B^o(0,r))$ for some $r > 0$ and prove that for any $\vec{x} \in B^o(0,r) \subset \mathbb{R}^n$

$$f(\vec{x}) = \sum_{|\alpha| \leq k} \frac{D^\alpha f(0)}{\alpha!} \vec{x}^\alpha + \sum_{|\alpha| = k+1} \frac{D^\alpha f(\vec{\xi})}{\alpha!} \vec{x}^\alpha,$$

where $\vec{\xi}$ lies on a line between 0 and $\vec{x}$. Students can assume Taylor's Theorem for a single variable. See, for example, Theorem 5.15 of Rudin, p. 110.