M611 Fall 2019 Assignment 6, due Friday Oct. 11

1. [10 pts] (**Evans 2.5.1.**) Write down an explict formula for a function u solving the initial-value problem

$$\begin{aligned} u_t + \vec{b} \cdot Du + cu &= 0 \quad \text{in } \mathbb{R}^n \times (0, \infty) \\ u &= g \quad \text{on } \mathbb{R}^n \times \{t = 0\}. \end{aligned}$$

Here $c \in \mathbb{R}$ and $\vec{b} \in \mathbb{R}^n$ are constants.

2. [10 pts] (**Evans 2.5.2.**) Prove that Laplace's equation $\Delta u = 0$ is rotation invariant; that is, if O is an orthogonal $n \times n$ matrix and we define

$$v(\vec{x}) := u(O\vec{x}), \quad \vec{x} \in \mathbb{R}^n,$$

then $\Delta v = 0$.

Notes. Recall that we say O is orthogonal if $O^T O = I$ (i.e., $O^{-1} = O^T$). Also, as motivation for the definition, recall that in $\mathbb{R}^2$ if the coordinates for a point P are (x_1, x_2) in one coordinate system and (y_1, y_2) in a coordinate system rotated counterclockwise by angle θ then

$$\vec{y} = O\vec{x},$$

where O is the rotation matrix

$$O = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

which is clearly orthogonal.

3. [10 pts] Recall from our discussion of physicality that the gravitational (force) field $\vec{\mathcal{F}}$ corresponding with mass distributed in space with density $\rho(\vec{x})$ is

$$\vec{\mathcal{F}} = -Du,$$

where the potential u solves Poisson's equation

$$-\Delta u = -4\pi G \rho(\vec{x}),$$

where G is Newton's gravitational constant. Use this formulation to derive an expression for the gravitational field $\vec{\mathcal{F}}$ associated with mass uniformly distributed over a ball $B(0, R) \subset \mathbb{R}^3$ with constant density ρ. Be sure to consider points both inside and outside $B(0, R)$.

Note. I suggest placing $\vec{x}$ on the vertical axis (without loss of generality) and using the law of cosines with the angle of inclination ϕ. Recall that if θ denotes your azimuthal angle, then the surface increment for a sphere is $dS = r^2 \sin\phi d\phi d\theta$.

4. [10 pts] Suppose $U \subset \mathbb{R}^n$ is open and $f \in C^1(U \times [0, \infty), \mathbb{R})$. For $B(\vec{x}_0, r) \subset U$, set

$$F(r) = \int_{B(\vec{x}_0, r)} f(\vec{x}, r) d\vec{x}.$$

Show that

$$F'(r) = \int_{B(\vec{x}_0, r)} f_r(\vec{x}, r) d\vec{x} + \int_{\partial B(\vec{x}_0, r)} f(\vec{x}, r) dS_x.$$