M611 Fall 2019, Assignment 7, due Friday Oct. 18

1. [10 pts] (**Evans 2.5.3.**) Modify the proof of the mean value formulas to show for $n \geq 3$ that

$$u(0) = \fint_{\partial B(0,r)} g dS + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \Big(\frac{1}{|\vec{x}|^{n-2}} - \frac{1}{r^{n-2}} \Big) f d\vec{x},$$

provided

$$\begin{aligned} -\Delta u &= f \quad \text{in } B^o(0,r) \\ u &= g \quad \text{in } \partial B(0,r). \end{aligned}$$

Note. Assume $f \in C(B(0,r))$ (the closed ball), and $g \in C(\partial B(0,r))$.

2. [10 pts] (**Evans 2.5.4.**) Give a direct proof that if $u \in C^2(U) \cap C(\bar{U})$ is harmonic within a bounded open set U, then

$$\max_{\bar{U}} u = \max_{\partial U} u.$$

(Hint: Define $u_\epsilon := u + \epsilon |\vec{x}|^2$ for $\epsilon > 0$, and show u_ϵ cannot attain its maximum over $\bar{U}$ at an interior point.)

3. [10 pts] (**Evans 2.5.5.**) We say $v \in C^2(\bar{U})$ is *subharmonic* if

$$-\Delta v \leq 0 \quad \text{in } U.$$

a. Prove for subharmonic v that

$$v(\vec{x}) \leq \fint_{B(\vec{x},r)} v d\vec{y} \quad \text{for all } B(\vec{x},r) \subset U.$$

b. Prove that therefore $\max_{\bar{U}} v = \max_{\partial U} v$.

c. Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume u is harmonic and $v := \phi(u)$. Prove v is subharmonic.

d. Prove $v := |Du|^2$ is subharmonic whenever u is harmonic.

4. [10 pts] (**Evans 2.5.6.**) Let U be a bounded, open subset of $\mathbb{R}^n$. Prove that there exists a constant C, depending only on U, such that

$$\max_{\bar{U}} |u| \leq C \Big(\max_{\partial U} |g| + \max_{\bar{U}} |f| \Big)$$

whenever u is a smooth solution of

$$\begin{aligned} -\Delta u &= f \quad \text{in } U \\ u &= g \quad \text{on } \partial U. \end{aligned}$$

(Hint: $-\Delta(u + \frac{|\vec{x}|^2}{2n}\lambda) \leq 0$, for $\lambda := \max_{\bar{U}} |f|$.)