Final Exam Topics

The final exam for M611 will be in the usual classroom BLOC 163, Wednesday, Dec. 11, 10:30 a.m.-12:30 p.m. The exam will cover the course material that was not covered on the midterm exam; in particular, Green's functions for Laplace's equation, and all of our material on the heat equation, the wave equation, and the method of characteristics. The exam will consist of four to six questions, with a mix of straightforward calculations and proofs. I will provide statements of the system of characteristic equations and of the d'Alembert, Poisson, and Kirchhoff solutions to the wave equation. You will need to bring your own paper.

1. Explicit calculations

Examples of problems we've solved by explicit calculation include the following:

- The derivation of Green's functions (by the method of images), and the application of Green's functions to solving Laplace's equation
- Multidimensional integration such as our evaluation of the volume of a heat ball
- Solutions of the wave equation on quarter planes
- Solutions of first order linear, quasilinear and nonlinear PDE by the method of characteristics

2. Proofs

Examples of topics from which proof-based problems may be taken include the following:

- Rigorous verification that solutions to Poisson's equation specified in terms of Green's functions are genuinely classical solutions
- Verify properties of the heat kernel and properties of solutions to the heat equation on $\mathbb{R}^n \times \mathbb{R}_+$
- Verify properties of the heat equation on U_T
- Derive properties of solutions of the wave equation, especially (though not only) in cases in which the properties can be obtained from the d'Alembert, Poisson, and Kirchhoff integral representations

Practice with the method of characteristics

1. (**Evans 3.5.5.**) Solve using characteristics:

a. $x_1 u_{x_1} + x_2 u_{x_2} = 2u; \quad u(x_1, 1) = g(x_1)$

c. $x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u; \quad u(x_1, x_2, 0) = g(x_1, x_2)$

b. $u u_{x_1} + u_{x_2} = 1; \quad u(x_1, x_1) = \frac{1}{2} x_1$

Solution. For (a), we begin by setting

$$z(s) = u(x^1(s), x^2(s)).$$

We compute

$$\frac{dz}{ds} = u_{x_1}(\vec{x}(s))\frac{dx^1}{ds} + u_{x_2}(\vec{x}(s))\frac{dx^2}{ds},$$

so that

$$\begin{aligned}
\frac{dx^1}{ds} &= x^1; \quad x^1(0) = x^1_0 \Rightarrow x^1(s) = x^1_0 e^s \\
\frac{dx^2}{ds} &= x^2; \quad x^2(0) = 1 \Rightarrow x^2(s) = e^s \\
\frac{dz}{ds} &= 2z; \quad z(0) = g(x^1_0) \Rightarrow z(s) = g(x^1_0) e^{2s}.
\end{aligned}$$

Solving for x^1_0 and s in terms of x^1 and x^2 we find

$$\begin{aligned}
x^1_0 &= \frac{x^1}{x^2} \\
e^s &= x^2,
\end{aligned}$$

so that

$$u(x_1, x_2) = g(\frac{x_1}{x_2}) x_2^2.$$

For (b), we set

$$\begin{aligned}
\frac{dx^1}{ds} &= x^1; \quad x^1(0) = x^1_0 \Rightarrow x^1(s) = x^1_0 e^s \\
\frac{dx^2}{ds} &= 2x^2; \quad x^2(0) = x^2_0 \Rightarrow x^2(s) = x^2_0 e^{2s} \\
\frac{dx^3}{ds} &= 1; \quad x^3(0) = 0 \Rightarrow x^3(s) = s \\
\frac{dz}{ds} &= 3z; \quad z(0) = g(x^1_0, x^2_0) \Rightarrow z(s) = g(x^1_0, x^2_0) e^{3s}.
\end{aligned}$$

We see that

$$\begin{aligned}
s &= x_3 \\
x^1_0 &= x_1 e^{-x_3} \\
x^2_0 &= x_2 e^{-2x_3},
\end{aligned}$$

so that

$$u(\vec{x}) = g(x_1 e^{-x_3}, x_2 e^{-2x_3}) e^{3x_3}.$$

For (c), we set

$$\begin{aligned}
\frac{dx^1}{ds} &= z(s); \quad x^1(0) = x_0^1 \Rightarrow x^1(s) = x_0^1 + \int_0^s z(\tau)d\tau \\
\frac{dx^2}{ds} &= 1; \quad x^2(0) = x_0^1 \Rightarrow x^2(s) = x_0^1 + s \\
\frac{dz}{ds} &= 1; \quad z(0) = \frac{1}{2}x_0^1 \Rightarrow z(s) = \frac{1}{2}x_0^1 + s.
\end{aligned}$$

We see that

$$x^1(s) = x_0^1 + \frac{s}{2}x_0^1 + \frac{s^2}{2}.$$

Now we notice $x_0^1 = x^2 - s$, so that

$$x^1 = (x^2 - s) + \frac{s}{2}(x^2 - s) + \frac{s^2}{2} = x^2 - s + \frac{s}{2}x^2.$$

Solving for s

$$s = \frac{x_2 - x_1}{1 - \frac{1}{2}x_2}.$$

This in turn gives

$$\begin{aligned}
x_0^1 &= x_2 - s = \frac{x_2 - \frac{1}{2}x_2^2 - (x_2 - x_1)}{1 - \frac{1}{2}x_2} \\
&= \frac{x_1 - \frac{1}{2}x_2^2}{1 - \frac{1}{2}x_2}.
\end{aligned}$$

We conclude

$$\begin{aligned}
u(x_1, x_2) &= \frac{1}{2}x_0^1 + s \\
&= \frac{\frac{1}{2}x_1 - \frac{1}{4}x_2^2 + x_2 - x_1}{1 - \frac{1}{2}x_2} \\
&= \frac{-\frac{1}{2}x_1 - \frac{1}{4}x_2^2 + x_2}{1 - \frac{1}{2}x_2}.
\end{aligned}$$

2. Find a solution to the PDE

$$\begin{aligned}
x_1 u_{x_1} + x_2 u_{x_2} + \frac{u_{x_1}^2 + u_{x_2}^2}{2} &= u \quad \text{in } \mathbb{R} \times \mathbb{R}_+ \\
u(x_1, 0) &= \frac{1 - x_1^2}{2}, \quad x_1 \in \mathbb{R}.
\end{aligned}$$

(Solutions to nonlinear PDE need not be unique; in particular, you need only find one solution.)

Solution. We begin by observing

$$F(\vec{p}, z, \vec{x}) = x_1 p_1 + x_2 p_2 + \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 - z,$$

so that

$$\begin{aligned} D_p F &= (x_1 + p_1, x_2 + p_2) \\ F_z &= -1 \\ D_x F &= (p_1, p_2). \end{aligned}$$

We have

$$u_{x_1}(x_1, 0) = -x_1 \Rightarrow p_0^1 = -x_0^1.$$

For p_0^2, our equation gives the relation

$$x_0^1 p_0^1 + x_0^2 p_0^2 + \frac{(p_0^1)^2 + (p_0^2)^2}{2} = \frac{1}{2} - \frac{1}{2}(x_0^1)^2,$$

which (upon noting $x_0^2 = 0$) gives the equation

$$\frac{1}{2}(p_0^2)^2 = \frac{1}{2} \Rightarrow p_0^2 = \pm 1.$$

This is where the uniqueness comment comes in, and we *choose* to find the solution for which $p_0^2 = +1$. We obtain the system

$$\begin{aligned} \frac{dp^1}{ds} &= 0; \quad p^1(0) = -x_0^1 \\ \frac{dp^2}{ds} &= 0; \quad p^2(0) = 1 \\ \frac{dz}{ds} &= z + \frac{1}{2}(p^1)^2 + \frac{1}{2}(p^2)^2; \quad z(0) = \frac{1}{2} - \frac{1}{2}(x_0^1)^2 \\ \frac{dx^1}{ds} &= x^1 + p^1; \quad x^1(0) = x_0^1 \\ \frac{dx^2}{ds} &= x^2 + p^2; \quad x^2(0) = 0. \end{aligned}$$

Solving this, we find

$$\begin{aligned} p^1(s) &= -x_0^1 \\ p^2(s) &= 1 \\ z(s) &= e^s - \frac{1}{2}\Big((x_0^1)^2 + 1\Big) \\ x^1(s) &= x_0^1 \\ x^2(s) &= -1 + e^s. \end{aligned}$$

We see that

$$\begin{aligned} x_0^1 &= x_1 \\ e^s &= 1 + x_2, \end{aligned}$$

allowing us to write

$$u(x_1, x_2) = z(s) = (1 + x_2) - \frac{1}{2}\Big(x_1^2 + 1\Big) = -\frac{1}{2}x_1^2 + x_2 + \frac{1}{2}.$$