M612 Spring 2020 Assignment 1, due Fri. Jan. 24

1. [10 pts] In this problem we consider two natural ways in which the Fourier transform arises in separation of variables.

a. Show that if we solve the heat equation

$$\begin{aligned} u_t &= u_{xx} \quad \text{in } \mathbb{R} \times \mathbb{R}_+ \\ \lim_{x \to \pm\infty} u(x,t) &= \text{bounded for each } t \in \mathbb{R}_+ \\ u(x,0) &= g(x) \quad x \in \mathbb{R} \end{aligned}$$

by making the separation of variables ansatz $u(x,t) = v(t)\phi(x)$ we find that $u(x,t)$ can formally be written in the form

$$u(x,t) = \int_{-\infty}^{+\infty} e^{iyx} e^{-y^2 t} A(y) dy,$$

for some function $A(y)$ that is to be determined. (I.e., in modern notation $u(x,t)$ is the inverse Fourier transform of $\sqrt{2\pi} e^{-y^2 t} A(y)$, where t is regarded as a parameter.) Use the (distributional) identity

$$\int_{-\infty}^{+\infty} e^{ix(y-\xi)} dx = 2\pi \delta(y - \xi)$$

to identify $A(y)$.

b. The Fourier series for a function $f \in C([-L, L])$ can be written as

$$f(x) = \sum_{k=-\infty}^{+\infty} c_k e^{i \frac{k\pi x}{L}},$$

where the expansion coefficients are given by

$$c_k = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i \frac{k\pi x}{L}} dx.$$

Set $\omega_k := \frac{k\pi}{L}$ (the *wave number*) and $\Delta\omega = \omega_{k+1} - \omega_k = \frac{\pi}{L}$, and show formally that the Fourier transform and its inverse are obtained in the limit as $L \to \infty$.

2. [10 pts] Solve the following:

a. Compute the Fourier transform of

$$f(x) = e^{-a|x|}$$

for $a > 0$ and $x \in \mathbb{R}$.

b. Use Part (a) to derive the identity

$$e^{-a|x|} = \frac{2a}{\sqrt{2\pi}} \int_0^{\infty} e^{-a^2 s} \frac{1}{\sqrt{2s}} e^{-\frac{x^2}{4s}} ds.$$

c. Use Part (b) to compute the Fourier transform of

$$f(x) = e^{-a|\vec{x}|}$$

for $a > 0$ and $\vec{x} \in \mathbb{R}^n$.

3. [10 pts] Use the method of Fourier transforms to solve Laplace's equation on the half-space

$$\begin{aligned} \Delta u &= 0; \quad \text{in } \mathbb{R}^n_+ \\ u &= g; \quad \text{on } \partial\mathbb{R}^n_+, \end{aligned}$$

where $g \in L^2(\mathbb{R}^{n-1})$ and we take $u, \Delta u \in L^2_{\tilde{x}}(\mathbb{R}^{n-1})$ for each $x_n \geq 0$ as a boundary condition. (Here $\vec{x} = (\tilde{x}, x_n)$.) Check your answer in Evans on p. 37.

4. [10 pts] (**Evans problem 4.7.13.**) Show that we can construct an explicit solution of the initial-value problem

$$\begin{aligned} v_t - v_{zz} + (\beta(t) z v)_z + \frac{\sigma'(t)}{\sigma(t)} v &= 0 \quad \text{in } \mathbb{R} \times \mathbb{R}_+ \\ v(z,0) &= \frac{\delta(z)}{\sigma(0)}, \quad z \in \mathbb{R}, \end{aligned}$$

having the form

$$v(z,t) = \frac{1}{\sigma(t)} \frac{1}{(\pi\gamma(t))^{1/2}} e^{-\frac{z^2}{\gamma(t)}}; \quad (z \in \mathbb{R}, t > 0),$$

the function $\gamma(t)$ to be found. Substitute into the PDE and determine an ODE that γ should satisfy. What is the initial condition for this ODE?

Note. Do this by taking a Fourier transform, using the (distributional) relation

$$\hat{\delta}(y) = \frac{1}{\sqrt{2\pi}}.$$

Assume $\sigma \in C^1[0,\infty)$ and $\sigma(t) \geq \sigma_0 > 0$ for all $t \geq 0$. You will (probably) find that $\gamma(t)$ is determined during your calculation, and in this case you don't need to substitute back into your PDE to find it.