M612 Spring 2020 Assignment 2, due Fri. Jan. 31

1. [10 pts] In this problem, we collect two straightforward relations needed for lecture.

a. Show that if $0 < \gamma \le 1$ and $\vec{x}, \vec{y} \in \mathbb{R}^n$ then

$$\Big| |\vec{x}|^\gamma - |\vec{y}|^\gamma \Big| \le |\vec{x} - \vec{y}|^\gamma.$$

b. Show that if σ, β, and γ are multiindices with $\gamma \le \sigma \le \beta$ and $|\gamma| = 1$ then

$$\binom{\beta}{\sigma - \gamma} + \binom{\beta}{\sigma} = \binom{\beta + \gamma}{\sigma}.$$

2. [10 pts] The different function spaces that arise in the analysis of PDEs can be regarded as categories of regularity (i.e., smoothness). For example, if $U \subset \mathbb{R}^n$ denotes an open set, we can think of $C(\bar{U})$ as a sort of baseline case with minimal regularity. The spaces $C^k(\bar{U})$, $k = 1, 2, \dots$ clearly have regularity that increases with increasing k. To be precise, we make the following definition: Suppose X denotes a space of functions defined on $U = B(0, 1)$. We define the regularity of X as

$$\operatorname{Reg}(X) := \inf_{\delta \in \mathbb{R} \setminus \mathbb{Z}} \{\delta : |\vec{x}|^\delta \in X\}.$$

(In particular, notice that negative regularity is possible.) Compute the regularity of each of the following spaces:

(i) $C(\bar{U})$.

(ii) $C^k(\bar{U})$, $k = 1, 2, 3, \dots$

(iii) $L^p(U)$, $1 \le p \le \infty$.

(vi) $C^{0,\gamma}(\bar{U})$, $0 < \gamma \le 1$.

Note. Rademacher's Theorem (Theorem 5.8.6 in Evans) asserts that if u is Lipschitz continuous in U then u is differentiable a.e. in U. Does this make sense according to regularity?

3. [10 pts] (**Evans 5.10.1.**) Suppose $k \in \{0, 1, 2, \dots\}$ and $0 < \gamma \le 1$. Prove that $C^{k,\gamma}(\bar{U})$ is a Banach space.

4. [10 pts] For $0 < \beta < \gamma \le 1$ prove the interpolation inequality

$$\|u\|_{C^{0,\gamma}(\bar{U})} \le \|u\|_{C^{0,\beta}(\bar{U})}^{\frac{1-\gamma}{1-\beta}} \cdot \|u\|_{C^{0,1}(\bar{U})}^{\frac{\gamma-\beta}{1-\beta}}.$$

Note. Observe the corresponding regularity relation,

$$\operatorname{Reg}\,(C^{0,\gamma}) = \frac{1-\gamma}{1-\beta}\operatorname{Reg}\,(C^{0,\beta}) + \frac{\gamma-\beta}{1-\beta}\operatorname{Reg}\,(C^{0,1}).$$