M612 Spring 2020, Assignment 3, due Fri. Feb. 7

1. [10 pts] Compute the weak derivative, if it exists, of

$$u(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational,} \end{cases}$$

on $U = (0, 1)$.

2. [10 pts] Determine whether or not

$$f(\vec{x}) = \text{sgn } (x_1)$$

is weakly differentiable on $U = B^o(0, 1) \subset \mathbb{R}^n$, and justify your claim. Show that f belongs to a space with 0 regularity, but that given any $r < 0$, we can find a space with regularity r that does not contain f. (Here, you don't have to think of n as fixed.)

3. [10 pts] For each $n \in \{2, 3, 4, \dots\}$, find the values $l \in \mathbb{R}$ so that

$$u(\vec{x}) = \Big| \ln |\vec{x}| \Big|^l$$

is weakly differentiable on $U = B^o(0, \frac{1}{2}) \subset \mathbb{R}^n$.

b. Find the values $l \in \mathbb{R}$ for which $u(\vec{x})$ is in $W^{1,p}(U)$. Take care with the case $n = p$.

c. We saw in class that for $n = 2$ $\text{Reg}(H^1(U)) = 0$. Show, however, that $H^1(U)$ is not a subset of $C(\bar{U})$ for $n = 2$.

4. [10 pts] (**Evans 5.10.6.**) Assume U is bounded and $U \subset\subset \cup_{i=1}^N V_i$. Show there exist C^∞ functions ζ_i $(i = 1, \dots, N)$ such that

$$\begin{aligned} 0 \le \zeta_i \le 1, \quad \text{spt } \zeta_i \subset V_i \quad (i = 1, \dots, N) \\ \sum_{i=1}^N \zeta_i = 1, \quad \text{on } U. \end{aligned}$$

The functions $\{\zeta_i\}_{i=1}^N$ form a *partition of unity.*