M612 Spring 2020 Assignment 5, due Fri., Feb. 21

1. [10 pts] Let $I \subset \mathbb{R}$ denote some open interval containing 0. Show that $Tf = f(0)$ is not a bounded linear functional on the space of continuous functions measured with the $L^2(I)$ norm, but is a bounded linear functional if measured using the uniform norm. In the latter case, the Hahn-Banach Theorem allows you to extend T to a bounded linear functional on $L^\infty(I)$. What does this say about $L^\infty(I)^*$?

2. [10 pts] We saw in class that if $u^* \in L^p(U)^*$ for some $1 \leq p < \infty$, then there exists $v \in L^q(U)$ so that

$$\langle u^*, u \rangle = \int_U uv d\vec{x}$$

for all $u \in L^p(U)$, and moreover that the map $u^* \mapsto v$ is an isometric isomorphism, so that $L^p(U)^* \overset{\text{i.i.}}{=} L^q(U)$. Show that in this setting (i.e., with $X = L^p(U)$), and with the additional assumption that $p > 1$, the canonical injection J (as defined in class) is surjective.

3. [10 pts] Let X, Y, and Z denote Banach spaces.

a. Show that the sum of two compact operators $A : X \to Y$ and $B : X \to Y$ is compact.

b. Show that if $A : X \to Y$ and $B : Y \to Z$ then:

(i) If A is bounded and B is compact then BA is compact.

(ii) If A is compact and B is bounded then BA is compact.

4. [10 pts] Let H denote a real Hilbert space with inner product $(\cdot, \cdot)$.

a. Show that if $u_k \rightharpoonup u$ in H then u is uniquely determined.

b. Show that if a linear operator $K : H \to H$ is compact and $u_k \rightharpoonup u$ then $Ku_k \to Ku$.