M612 Spring 2020 Assignment 6, due Fri., Feb. 28

1. [10 pts] Let $U \subset \mathbb{R}^n$ be open and let u be a measurable function defined on U. Set

$$U_{u,t} := \{\vec{x} \in U : |u(\vec{x})| > t\},$$

and the distribution function of u

$$\delta_u(t) := \mu(U_{u,t});$$

i.e., the Lebesgue measure of $U_{u,t}$. We say u is in the space weak-L^p provided

$$[u]_{L^p} := \Big(\sup_{t>0} t^p \delta_u(t) \Big)^{1/p}$$

is finite. Compute Reg (weak-L^p).

2. [10 pts] (**Evans 5.10.8.**) Let U be bounded, with a C^1 boundary. Show that a "typical" function $u \in L^p(U)$ $(1 \leq p < \infty)$ does not have a trace on ∂U. More precisely, prove there does not exist a bounded linear operator

$$T : L^p(U) \to L^p(\partial U)$$

such that $Tu = u|_{\partial U}$ whenever $u \in C(\bar{U}) \cap L^p(U)$.

3. [10 pts] (**Evans 5.10.9.**) Integrate by parts to prove the interpolation inequality:

$$\|Du\|_{L^2} \leq C \|u\|_{L^2}^{1/2} \|D^2 u\|_{L^2}^{1/2}$$

for all $u \in C_c^\infty(U)$. Assume U is bounded, ∂U is smooth, and prove this inequality if $u \in H^2(U) \cap H_0^1(U)$. (Hint: Take sequences $\{v_k\}_{k=1}^\infty \subset C_c^\infty(U)$ converging to u in $H_0^1(U)$ and $\{w_k\}_{k=1}^\infty \in C^\infty(\bar{U})$ converging to u in $H^2(U)$.)

4. [10 pts] (**Evans 5.10.10.**) Answer the following:

a. Integrate by parts to prove

$$\|Du\|_{L^p} \leq C \|u\|_{L^p}^{1/2} \|D^2 u\|_{L^p}^{1/2}$$

for $2 \leq p < \infty$ and for all $u \in C_c^\infty(U)$. Hint:

$$\int_U |Du|^p d\vec{x} = \sum_{i=1}^n \int_U u_{x_i} u_{x_i} |Du|^{p-2} d\vec{x}.$$

b. Prove

$$\|Du\|_{L^{2p}} \leq C \|u\|_{L^\infty}^{1/2} \|D^2 u\|_{L^p}^{1/2}$$

for $1 \leq p < \infty$ and all $u \in C_c^\infty(U)$.