M612 Spring 2020 Assignment 8, due Fri., Apr. 3

1. [10 pts] (**Evans 5.10.13.**) Give an example of an open set $U \subset \mathbb{R}^n$ and a function $u \in W^{1,\infty}(U)$, such that u is *not* Lipschitz continuous on U. (Hint: Take U to be the open unit disk in $\mathbb{R}^2$, with a slit removed.)

Note. At the beginning of this problem set (in Evans), Evans states that his boundaries will all be smooth, but it's clear from his hint (and Theorem 5.8.4) that he intends to relax that here. Also, if you decide to use a different set from the one in Evans' hint, take it be be connected. (I.e., this is too easy for sets that are not connected.)

2. [10 pts] (**Evans 5.10.17.**) (Chain rule) Assume $F : \mathbb{R} \to \mathbb{R}$ is C^1, with F' bounded. Suppose U is bounded and $u \in W^{1,p}(U)$ for some $1 \leq p \leq \infty$. Show

$$v := F(u) \in W^{1,p}(U) \quad \text{and} \quad v_{x_i} = F'(u)u_{x_i} \quad (i = 1, \dots, n).$$

3. [10 pts] (**Evans 5.10.18.**) Assume $1 \leq p \leq \infty$ and U is bounded.

a. Prove that if $u \in W^{1,p}(U)$, then $|u| \in W^{1,p}(U)$.

b. Prove $u \in W^{1,p}(U)$ implies $u^+, u^- \in W^{1,p}(U)$, and

$$Du^+ = \begin{cases} Du & \text{a.e. on } \{u > 0\} \\ 0 & \text{a.e. on } \{u \leq 0\} \end{cases}$$

$$Du^- = \begin{cases} 0 & \text{a.e. on } \{u \geq 0\} \\ -Du & \text{a.e. on } \{u < 0\}. \end{cases}$$

(Hint: $u^+ = \lim_{\epsilon \to 0} F_\epsilon(u)$, for

$$F_\epsilon(z) := \begin{cases} (z^2 + \epsilon^2)^{1/2} - \epsilon & \text{if } z \geq 0 \\ 0 & \text{if } z < 0.) \end{cases}$$

c. Prove that if $u \in W^{1,p}(U)$, then

$$Du = 0 \,\text{a.e.} \quad \text{on the set } \{u = 0\}.$$

4. [10 pts] (**Evans 6.6.1.**) Consider Laplace's equation with potential function c :

$$-\Delta u + cu = 0, \tag{*}$$

and the divergence structure equation

$$-\text{div } (aDv) = 0, \tag{**}$$

where the function a is positive.

a. Show that if u solves (*) and $w > 0$ also solves (*), then $v := u/w$ solves (**) for $a := w^2$.

b. Conversely, show that if v solves (**), then $u := va^{1/2}$ solves (*) for some potential c.