Second Order Elliptic PDE: The Fredholm Alternative, I

MATH 612, Texas A&M University

Spring 2020

Adjoint Operators on Hilbert Spaces

Our next goal will be to develop an existence theorem known as the Fredholm Alternative. For this, we'll need some background on adjoint operators.

Definition. Suppose H denotes a real Hilbert space, and $A : H \to H$ is a bounded linear operator. For each fixed $v \in H$, the map $u \mapsto (Au, v)$ is a bounded linear operator, so the Riesz Representation Theorem guarantees that there exists a unique $w \in H$ so that

$$(Au, v) = (u, w)$$

for all $u \in H$. This specifies a well-defined map that takes $v \in H$ as input and returns the corresponding $w \in H$. We typically denote this map A^*, and refer to A^* as the adjoint of A. I.e., A^* is defined so that

$$(Au, v) = (u, A^* v),$$

for all $u, v \in H$.

Adjoint Operators on Hilbert Spaces

Notes. 1. We'll check in the homework that if $A : H \to H$ is a bounded linear operator, then so is $A^* : H \to H$, and $\|A^*\| = \|A\|$.

2. If $A^* = A$, we say that A is self-adjoint.

Theorem A.D.4. If $K : H \to H$ is a compact linear operator, then so is $K^* : H \to H$.

Proof. Let $\{u_k\}_{k=1}^{\infty}$ be a bounded sequence in H, and recall that to show that K^* is compact we need to show that there exists a subsequence $\{u_{k_j}\}_{j=1}^{\infty} \subset \{u_k\}_{k=1}^{\infty}$ so that the sequence $\{K^* u_{k_j}\}_{j=1}^{\infty}$ converges in H.

According to Theorem A.D.3 from our discussion of functional analysis, there exists a subsequence $\{u_{k_j}\}_{j=1}^{\infty} \subset \{u_k\}_{k=1}^{\infty}$ so that $u_{k_j} \rightharpoonup u$ in H. (Notice that the convergence is weak.)

Adjoint Operators on Hilbert Spaces

We can compute

$$\begin{aligned}\|K^* u_{k_j} - K^* u\|^2 &= (K^* u_{k_j} - K^* u, K^*(u_{k_j} - u)) \\ &= (KK^* u_{k_j} - KK^* u, (u_{k_j} - u)).\end{aligned}$$

Recall that for any $f \in H^*$, we can use the Riesz Representation Theorem to find $w \in H$ so that $\langle f, v \rangle = (w, v)$ for all $v \in H$. This allows us to write

$$\begin{aligned}&\langle f, K^*(u_{k_j} - u) \rangle \\ &= (w, K^*(u_{k_j} - u)) = (Kw, u_{k_j} - u) \stackrel{j\to\infty}{\to} 0,\end{aligned}$$

where the limit follows from the weak convergence of $\{u_{k_j}\}_{j=1}^{\infty}$. We conclude that $K^* u_{k_j} \rightharpoonup K^* u$, and we recall from a homework problem that this implies $KK^* u_{k_j} \to KK^* u$ (strong convergence).

Adjoint Operators on Hilbert Spaces

Returning to the relation

$$\|K^* u_{k_j} - K^* u\|^2 = (KK^* u_{k_j} - KK^* u, (u_{k_j} - u)),$$

we can now write,

$$\begin{aligned} \|K^* u_{k_j} - K^* u\|^2 &= |(KK^* u_{k_j} - KK^* u, (u_{k_j} - u))| \\ &\le \|KK^* u_{k_j} - KK^* u\| \|u_{k_j} - u\|. \end{aligned}$$

Since $\{u_{k_j}\}_{j=1}^{\infty}$ is weakly convergent, $\|u_{k_j} - u\|$ is bounded, and we've seen that

$$\lim_{j \to \infty} \|KK^* u_{k_j} - KK^* u\| = 0.$$

We conclude that $K^* u_{k_j} \to K^* u$ as $j \to \infty$, which is what we needed to show to establish that K^* is compact. $\square$

The Fredholm Alternative for Compact Operators

Suppose X denotes a real Banach space and $A : X \to X$ denotes a bounded linear operator. We'll denote by $N(A)$ the kernel of A, and we'll denote by $R(A)$ the range of A.

Theorem A.D.5. (Fredholm Alternative for Compact Operators) Suppose H is a real Hilbert space, and $K : H \to H$ is a compact linear operator. Then:

(i) $N(I-K)$ is finite dimensional.

(ii) $R(I-K)$ is closed.

(iii) $R(I-K) = N(I-K^*)^\perp$.

(iv) $N(I-K) = \{0\} \iff R(I-K) = H$.

(v) $\dim N(I-K) = \dim N(I-K^*)$.

The Fredholm Alternative for Compact Operators

Notice particularly that Item (iv) provides a charaterization of the solvability of the operator equation

$$(I - K)u = f, \qquad (*)$$

for $f \in H$. It asserts that exactly one of the following two statements must be true:

for each $f \in H$ equation (*) has a unique solution $u \in H$

or

the equation $(I - K)u = 0$ has one or more solutions $u \neq 0$.

This statement is referred to as the Fredholm Alternative.

If the latter case of the alternative holds, then by (iii) (*) has a solution iff $f \in N(I - K^*)^{\perp}$.

The Formal Adjoint Operator for L

Let's return to our divergence-form operator

$$Lu = -\sum_{i,j=1}^{n} (a^{ij} u_{x_i})_{x_j} + \sum_{i=1}^{n} b^i u_{x_i} + cu.$$

Formally (i.e., assuming as much smoothness as we need on both u and the coefficients), we can multiply both sides by $v \in C_c^\infty(U)$ and integrate to see that

$$(Lu, v) = \int_U \Big\{ -\sum_{i,j=1}^{n} (a^{ij} u_{x_i})_{x_j} + \sum_{i=1}^{n} b^i u_{x_i} + cu \Big\} v d\vec{x}$$

$$\overset{\text{parts}}{=} \int_U u \Big\{ -\sum_{i,j=1}^{n} (a^{ij} v_{x_j})_{x_i} - \sum_{i=1}^{n} (b^i v)_{x_i} + cv \Big\} d\vec{x}.$$

The Formal Adjoint Operator for L

We would like to introduce an operator L^* so that

$$(Lu, v) = (u, L^* v),$$

and we see that in our formal setting

$$L^* v = -\sum_{i,j=1}^{n} (a^{ij} v_{x_j})_{x_i} - \sum_{i=1}^{n} (b^i v)_{x_i} + cv,$$

works. Since L is not bounded, it doesn't satisfy our framework above. For now, we'll refer to L^* as the formal adjoint of L.

Notice from the previous slide that formally (upon integrating by parts)

$$B[u, v] = (Lu, v) = (u, L^* v) = (L^* v, u) =: B^*[v, u]$$

This motivates the definitions on the next slide.

The Formal Adjoint Operator for L

Definitions.

(i) We define the adjoint bilinear form of $B[u, v]$ by the relation

$$B^*[v, u] = B[u, v].$$

(ii) For $f \in H^{-1}(U)$, we say that $v \in H_0^1(U)$ is a weak solution of the adjoint problem

$$\begin{aligned} L^* v &= f; \quad \text{in } U \\ v &= 0; \quad \text{on } \partial U, \end{aligned}$$

provided

$$B^*[v, u] = \langle f, u \rangle$$

for all $u \in H_0^1(U)$.

The Fredholm Alternative for Elliptic Operators

Theorem 6.2.4. Suppose $U \subset \mathbb{R}^n$ is open and bounded, a^{ij}, b^i, $c \in L^\infty(U)$ ($\forall\, i, j \in \{1, 2, \dots n\}$), and L is uniformly elliptic. Then:

(i) Precisely one of the following two statements must be true:

(α) For each $f \in L^2(U)$, the PDE

$$\begin{aligned} Lu &= f; \quad \text{in } U \\ u &= 0; \quad \text{on } \partial U, \end{aligned}$$

has a unique weak solution $u \in H_0^1(U)$; **or**

(β) The homogeneous PDE

$$\begin{aligned} Lu &= 0; \quad \text{in } U \\ u &= 0; \quad \text{on } \partial U, \end{aligned}$$

has a non-trivial weak solution $u \in H_0^1(U)$.

The Fredholm Alternative for Elliptic Operators

Theorem 6.2.4 cont.

(ii) In the event of (β), let N denote the solution space of the equation in (β), and let N^* denote the solution space of

$$\begin{aligned} L^* v &= 0; \quad \text{in } U \\ v &= 0; \quad \text{on } \partial U. \end{aligned}$$

Then $\dim N = \dim N^*$, and the mutual value is finite.

(iii) The equation in (α) has a weak solution $u \in H_0^1(U)$ if and only if

$$(f, v) = 0, \quad \forall\, v \in N^*.$$

I.e., if $f \in (N^*)^{\perp}$.