Second Order Elliptic PDE: The Fredholm Alternative, II

MATH 612, Texas A&M University

Spring 2020

The Fredholm Alternative for Elliptic Operators

Theorem 6.2.4. Suppose $U \subset \mathbb{R}^n$ is open and bounded, a^{ij}, b^i, $c \in L^\infty(U)$ $(\forall\, i, j \in \{1, 2, \dots n\})$, and L is uniformly elliptic. Then:

(i) Precisely one of the following two statements must be true:

(α) For each $f \in L^2(U)$, the PDE

$$\begin{aligned} Lu &= f; \quad \text{in } U \\ u &= 0; \quad \text{on } \partial U, \end{aligned}$$

has a unique weak solution $u \in H_0^1(U)$; **or**

(β) The homogeneous PDE

$$\begin{aligned} Lu &= 0; \quad \text{in } U \\ u &= 0; \quad \text{on } \partial U, \end{aligned}$$

has a non-trivial weak solution $u \in H_0^1(U)$.

The Fredholm Alternative for Elliptic Operators

Theorem 6.2.4 cont.

(ii) In the event of (β), let N denote the solution space of the equation in (β), and let N^* denote the solution space of

$$\begin{aligned} L^* v &= 0; \quad \text{in } U \\ v &= 0; \quad \text{on } \partial U. \end{aligned}$$

Then $\dim N = \dim N^*$, and the mutual value is finite.

(iii) The equation in (α) has a weak solution $u \in H_0^1(U)$ if and only if

$$(f, v) = 0, \quad \forall\, v \in N^*.$$

I.e., if $f \in (N^*)^\perp$.

Proof of Theorem 6.2.4

1. Our strategy will be to apply the Fredholm Alternative for compact operators, so the first thing we'll need to do is identify an appropriate compact operator. To this end, let $\gamma \geq 0$ be as in Theorem 6.2.2, so that for some $\beta > 0$ we have the lower bound

$$B[u,u] \geq \beta \|u\|^2_{H^1(U)} - \gamma \|u\|^2_{L^2(U)}.$$

Set

$$B_\gamma[u,v] := B[u,v] + \gamma(u,v),$$

and

$$L_\gamma u := Lu + \gamma u.$$

We saw in Theorem 6.2.3 that for each $g \in L^2(U)$ there exists a unique $u \in H^1_0(U)$ so that

$$B_\gamma[u,v] = (g,v), \quad \forall\, v \in H^1_0(U).$$

Proof of Theorem 6.2.4

We'll denote the corresponding map taking $g \in L^2(U)$ to $u \in H_0^1(U)$ by L_γ^{-1}. I.e., $L_\gamma^{-1} : L^2(U) \to H_0^1(U)$ is specified by $u = L_\gamma^{-1} g$ so that

$$B_\gamma[L_\gamma^{-1} g, v] = (g, v), \quad \forall\, v \in H_0^1(U).$$

Aside from a choice of scale, L_γ^{-1}, viewed as a map $L^2(U) \to L^2(U)$, will be the compact operator we're looking for.

2. Now, $u \in H_0^1(U)$ is a unique solution to the equation in (α) if and only if u uniquely solves

$$B[u, v] = (f, v), \quad \forall\, v \in H_0^1(U),$$

and this can be expressed in terms of $B_\gamma[u, v]$ as

$$B_\gamma[u, v] = (f + \gamma u, v), \quad \forall\, v \in H_0^1(U).$$

Proof of Theorem 6.2.4

In our notation from Step 1,

$$u = L_\gamma^{-1}(f + \gamma u) = L_\gamma^{-1} f + \gamma L_\gamma^{-1} u.$$

Let's set $h = L_\gamma^{-1} f \in H_0^1(U)$ and $K = \gamma L_\gamma^{-1}$. We see that $u \in H_0^1(U)$ is a unique solution to the equation in (α) if and only if u uniquely solves

$$u - Ku = h.$$

3. **Claim.** If we regard K as a map $K : L^2(U) \to L^2(U)$, then K is compact.

To see this, we first observe that for any $g \in L^2(U)$, the unique $u \in H_0^1(U)$ satisfying

$$B_\gamma[u, v] = (g, v), \quad \forall\, v \in H_0^1(U),$$

must also (by the coercivity of $B_\gamma[u, v]$) satisfy

$$\beta \|u\|_{H^1(U)}^2 \leq B_\gamma[u, u] = (g, u) \overset{c.s.}{\leq} \|g\|_{L^2(U)} \|u\|_{L^2(U)}$$

Proof of Theorem 6.2.4

We see that

$$\beta\|u\|_{H^1(U)}^2 \leq \|g\|_{L^2(U)}\|u\|_{L^2(U)} \leq \|g\|_{L^2(U)}\|u\|_{H^1(U)},$$

and so

$$\|u\|_{H^1(U)} \leq \frac{1}{\beta}\|g\|_{L^2(U)}.$$

We can write this as

$$\|L_\gamma^{-1}g\|_{H^1(U)} \leq \frac{1}{\beta}\|g\|_{L^2(U)},$$

or equivalently

$$\|Kg\|_{H^1(U)} \leq \frac{\gamma}{\beta}\|g\|_{L^2(U)}.$$

We see that $K : L^2(U) \to H_0^1(U)$ is a bounded linear operator.

Proof of Theorem 6.2.4

Next, since $\mathrm{Reg}(L^2) < \mathrm{Reg}(H^1)$ (strictly), we have the compact embedding $H_0^1(U) \subset\subset L^2(U)$. I.e., bounded subsets of $H_0^1(U)$ are precompact in $L^2(U)$.

Since K is bounded, it maps bounded subsets of $L^2(U)$ into bounded subsets of $H_0^1(U)$, and by compact embedding, bounded subsets of $H_0^1(U)$ are precompact in $L^2(U)$. We see that K maps bounded subsets of $L^2(U)$ into precompact subsets of $L^2(U)$, and so K is compact.

4. The Fredholm alternative for compact operators asserts: either

(A) for each $h \in L^2(U)$, $u - Ku = h$ has a unique solution $u \in L^2(U)$; **or**

(B) $u - Ku = 0$ has a non-trivial solution $u \in L^2(U)$.

Proof of Theorem 6.2.4

Notice that in the current setting, with $h \in H_0^1(U)$ and $K : L^2(U) \to H_0^1(U)$, the solutions described in (A) and (B) are all in $H_0^1(U)$.

Also, keep in mind that (α) and (β) are necessarily mutually exclusive, so we only need to check that at least one of them always holds (i.e., it's clear that they cannot both hold).

We've already seen that if (A) holds then (α) will hold (and so (β) will not). Notice that this necessarily includes the case $\gamma = 0$, because (B) cannot hold in that case (since $K = \gamma L_\gamma^{-1}$, if $\gamma = 0$ the equation in (B) is $u = 0$).

Likewise, if (B) holds, then the non-trivial solutions to $u - Ku = 0$ will satisfy the homogeneous equation in (β), and so (β) will hold (and so (α) will not). This completes the proof of (i).

Proof of Theorem 6.2.4

For (ii), if (β) holds, then (B) must hold, and N (the solution space for the homogeneous equation in (β)) will be the solution space of the homogenous problem $u - Ku = 0$.

According to Theorem A.D.5 (the Fredholm Alternative for compact operators, discussed in the previous lecture), dim N is finite (Item (i) of Theorem A.D.5) and dim $N =$ dim N^*, where N^* denotes the dimension of the solution space of the homogeneous problem $u - K^*u = 0$ (Item (v) of Theorem A.D.5).

In order to conclude (ii), we need to be clear about the relationship between K^* and the formal adjoint L^*.

Proof of Theorem 6.2.4

Claim. If L^* denotes the formal adjoint of L, and we set $L^*_\gamma := L^* + \gamma$, then $K^* = \gamma L^{*-1}_\gamma$.

To see this, we first recall from Step 1 that for any $g \in L^2(U)$ we can find a unique $u \in H^1_0(U)$ so that

$$B_\gamma[u, v] = (g, v), \quad \forall\, v \in H^1_0(U), \quad \text{i.e., } u = L^{-1}_\gamma g.$$

Proceeding in exactly the same way, we can show that given any $\tilde{g} \in L^2(U)$ we can find a unique $\tilde{u} \in H^1_0(U)$ so that

$$B^*_\gamma[\tilde{u}, \tilde{v}] = (\tilde{g}, \tilde{v}), \quad \forall \tilde{v} \in H^1_0(U), \quad \text{i.e., } \tilde{u} = L^{*-1}_\gamma \tilde{g}.$$

Recalling the defining relation $B^*_\gamma[\tilde{u}, \tilde{v}] := B_\gamma[\tilde{v}, \tilde{u}]$, we can take $\tilde{v} = u$ and $v = \tilde{u}$ to see that

$$(\tilde{g}, u) = B^*_\gamma[\tilde{u}, u] = B_\gamma[u, \tilde{u}] = (g, \tilde{u}).$$

Proof of Theorem 6.2.4

From the previous slide,

$$(\tilde{g}, u) = B_\gamma^*[\tilde{u}, u] = B_\gamma[u, \tilde{u}] = (g, \tilde{u}).$$

We have $u = L_\gamma^{-1} g$ and $\tilde{u} = L_\gamma^{*-1} \tilde{g}$ so that

$$(\tilde{g}, L_\gamma^{-1} g) = (g, L_\gamma^{*-1} \tilde{g}) \overset{\text{sym}}{=} (L_\gamma^{*-1} \tilde{g}, g), \quad \forall\, g, \tilde{g} \in L^2(U).$$

Recalling that $K = \gamma L_\gamma^{-1}$, we see that $K^* = \gamma L_\gamma^{*-1}$, establishing the claim.

It follows, precisely as for L_γ and K, that N^* (the solution space for the adjoint homogeneous problem in (ii)) will be the solution space for $u - K^* u = 0$. This completes the proof of (ii).

Proof of Theorem 6.2.4

5. For (iii), in the case that (α) holds, we have $N^* = \{0\}$ (so that $(N^*)^\perp = L^2(U)$), and so (iii) simply agrees with (α).

In the case that (β) holds, we must have that (B) holds, and from Theorem A.D.5 we know that we can solve

$$u - Ku = h$$

if and only if

$$h \in N(I - K^*)^\perp;$$

that is, if and only if $(h, v) = 0$ for all v satisfying

$$v - K^* v = 0.$$

Proof of Theorem 6.2.4

Recalling that $h = L_\gamma^{-1} f$, we can compute

$$0 = (h, v) = (L_\gamma^{-1} f, v) = \frac{1}{\gamma}(Kf, v)$$
$$= \frac{1}{\gamma}(f, K^* v) = \frac{1}{\gamma}(f, v).$$

I.e., $f \in (N^*)^\perp$ if and only if $h \in (N^*)^\perp$, so the equation in (α) has a solution if and only if $f \in (N^*)^\perp$, and this is (iii).

Here, we've used the fact that in case (B) we have $\gamma > 0$, and in the final equality we used $K^* v = v$. $\square$