Second Order Elliptic PDE: Higher Interior Regularity and Boundary Regularity

MATH 612, Texas A&M University

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Higher Interior Regularity

If we assume additional smoothness on the coefficients of our elliptic operator L, and likewise assume f is in a higher regularity space, then we can conclude additional local regularity on our weak solution u.

We'll summarize two theorems along these lines, and then move on to boundary regularity.

Higher Interior Regularity

Theorem 6.3.2. Suppose $U \subset \mathbb{R}^n$ is open and bounded, a^{ij}, b^i, $c \in C^{m+1}(U)$ ($\forall\, i, j \in \{1, 2, \dots n\}$) for some $m \in \{0, 1, 2, \dots\}$, and L is uniformly elliptic. Also, suppose $f \in H^m(U)$, and $u \in H^1(U)$ (not necessarily $H_0^1(U)$) is a weak solution of

$$Lu = f \quad \text{in } U.$$

Then $u \in H^{m+2}_{\mathrm{loc}}(U)$, and for each $V \subset\subset U$, there exists a constant C, depending only on V, U, m, and the coefficients of L, so that

$$\|u\|_{H^{m+2}(V)} \leq C(\|f\|_{H^m(U)} + \|u\|_{L^2(U)}).$$

Higher Interior Regularity

Notes. 1. The proof proceeds by induction on m, noting that Theorem 6.3.1 is (with minor adjustments) the case $m = 0$.

2. Since Reg $(H^{m+2}) = m + 2 - \frac{n}{2}$, we can conclude that for each $V \subset\subset U$, $u \in C^{\ell^*,\gamma^*}(\bar{V})$, where we recall that

$$\ell^* = m + 2 - [\frac{n}{2}] - 1,$$

and

$$\gamma^* = \begin{cases} [\frac{n}{2}] + 1 - \frac{n}{2} & \text{if } \frac{n}{2} \text{ is not an integer} \\ \text{any value } \in (0,1) & \text{if } \frac{n}{2} \text{ is an integer.} \end{cases}$$

Since $V \subset\subset U$ is arbitrary, $u \in C^{\ell^*}(U)$. For example, if $n = 3$ and $m = 2$, then for each $V \subset\subset U$, $u \in C^{2,\frac{1}{2}}(\bar{V})$, so u is a classical solution (i.e., $u \in C^2(U)$).

Higher Interior Regularity

Theorem 6.3.3. Suppose $U \subset \mathbb{R}^n$ is open and bounded, a^{ij}, b^i, $c \in C^\infty(U)$ ($\forall\, i, j \in \{1, 2, \ldots n\}$), and L is uniformly elliptic. Also, suppose $f \in C^\infty(U)$, and $u \in H^1(U)$ (not necessarily $H^1_0(U)$) is a weak solution of

$$Lu = f \quad \text{in } U.$$

Then $u \in C^\infty(U)$.

Note. Notice that $f \in C^\infty(U) \implies f \in H^\infty(W)$ for any $W \subset\subset U$, and this allows us to apply Theorem 6.3.2 (with W replacing U) for all $m \in \{0, 1, 2, \ldots\}$.

Boundary Regularity

In order to extend interior regularity to the boundary (i.e., to remove the local nature of the last three theorems), we need to assume some smoothness on ∂U.

Theorem 6.3.4. Suppose $U \subset \mathbb{R}^n$ is open and bounded with C^2 boundary, $a^{ij} \in C^1(\bar{U})$, b^i, $c \in L^\infty(U)$ $(\forall\, i,j \in \{1,2,\ldots n\})$, and L is uniformly elliptic. Also, suppose $f \in L^2(U)$, and $u \in H_0^1(U)$ is a weak solution of

$$\begin{aligned} Lu &= f \quad \text{in } U \\ u &= 0, \quad \text{on } \partial U. \end{aligned} \tag{*}$$

Then $u \in H^2(U)$, and there exists a constant C, depending only on U and the coefficients of L, so that

$$\|u\|_{H^2(U)} \leq C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}).$$

Boundary Regularity

Note. If $u \in H_0^1(U)$ is the unique weak solution of (*), then according to Theorem 6.2.6 (boundedness of the resolvent), we have

$$\|u\|_{L^2(U)} \leq \tilde{C}\|f\|_{L^2(U)},$$

for some constant $\tilde{C}$ that depends only on U and the coefficients of L. It follows that there exists a constant $\tilde{\tilde{C}}$, depending only on U and the coefficients of L, so that

$$\|u\|_{H^2(U)} \leq \tilde{\tilde{C}}\|f\|_{L^2(U)}.$$

Proof of Theorem 6.3.4

1. We'll start by working with a locally flat boundary and show in Step 6 how to map the general case into that setting. We'll work locally near a point $\vec{x}_0 \in \partial\mathbb{R}^n_+$, and for notational convenience, we'll shift coordinates so that $\vec{x}_0 = 0$.

We set

$$U := B^o(0,1) \cap \mathbb{R}^n_+ \quad \text{(i.e., this is } U \text{ in the first steps)}$$
$$V := B^o(0,\frac{1}{2}) \cap \mathbb{R}^n_+,$$

and we'll introduce a cut-off function $\zeta \in C_c^\infty(\mathbb{R}^n)$ so that

$$\zeta(\vec{x}) = \begin{cases} 1 & \vec{x} \in B(0,\frac{1}{2}) \\ \in [0,1] & \vec{x} \in B(0,\frac{3}{4}) \backslash B(0,\frac{1}{2}) \\ 0 & \vec{x} \in \mathbb{R}^n \backslash B(0,\frac{3}{4}). \end{cases}$$

Proof of Theorem 6.3.4

2. Since $u \in H_0^1(U)$ is a weak solution of (*), we have

$$B[u, v] = (f, v), \quad \forall\, v \in H_0^1(U),$$

and precisely as in Step 2 of the proof of Theorem 6.3.1 we can express this equation as

$$\int_U \sum_{i,j=1}^{n} a^{ij} u_{x_i} v_{x_j} d\vec{x} = (\tilde{f}, v), \quad \forall\, v \in H_0^1(U), \qquad (**)$$

where

$$\tilde{f} := f - \sum_{i=1}^{n} b^i u_{x_i} - cu \in L^2(U).$$

Proof of Theorem 6.3.4

3. Fix any $k \in \{1, 2, \ldots, n-1\}$ and any $0 < |h| < \frac{1}{8}$, and as in the proof of Theorem 6.3.1, set

$$v = -D_k^{-h}(\zeta^2 D_k^h u).$$

We would like to substitute v into (**) as in the proof of Theorem 6.3.1, but we need to verify that in the current setting we have $v \in H_0^1(U)$. For this, notice that

$$\begin{aligned} v(\vec{x}) &= -D_k^{-h}\left(\zeta(\vec{x})^2 \frac{u(\vec{x}+h\hat{e}_k) - u(\vec{x})}{h}\right) \\ &= -\left\{\zeta(\vec{x}-h\hat{e}_k)^2 \frac{u(\vec{x}) - u(\vec{x}-h\hat{e}_k)}{-h^2} - \zeta(\vec{x})^2 \frac{u(\vec{x}+h\hat{e}_k) - u(\vec{x})}{-h^2}\right\} \\ &= \frac{1}{h^2}\left\{\zeta(\vec{x}-h\hat{e}_k)^2 (u(\vec{x}) - u(\vec{x}-h\hat{e}_k)) - \zeta(\vec{x})^2 (u(\vec{x}+h\hat{e}_k) - u(\vec{x}))\right\}. \end{aligned}$$

Proof of Theorem 6.3.4

Since $k \in \{1, 2, \ldots, n-1\}$, the points $\vec{x} \pm h\hat{e}_k$ never leave $\overline{\mathbb{R}^n_+}$ (for $\vec{x} \in \overline{\mathbb{R}^n_+}$). Since ζ is 0 in $\mathbb{R}^n \backslash B(0, \frac{3}{4})$, it's clear that $v(\vec{x}) = 0$ for $\vec{x}$ near $\partial B(0,1) \cap \mathbb{R}^n_+$, and since $u = 0$ in the trace sense on $B(0,1) \cap \partial \mathbb{R}^n_+$, we see that $v = 0$ in the trace sense on ∂U.

Otherwise, v is a sum of terms in $H^1(U)$ (as in the proof of Theorem 6.3.1), so $v \in H^1_0(U)$.

We're now justified in substituting v into (**), and as in the proof of Theorem 6.3.1, we'll denote the left-hand side A and the right-hand side B.

Proof of Theorem 6.3.4

4. Proceeding as in Steps 4-6 of the proof of Theorem 6.3.1, we obtain the inequality

$$\int_V |D_k^h Du|^2 d\vec{x} \leq C_1 \int_U f^2 + u^2 + |Du|^2 d\vec{x}, \quad k \in \{1, 2, \ldots, n-1\},$$

for some constant C_1 (which was C_8 in the proof of Theorem 6.3.1). From another slight restatement of Theorem 5.8.3 (ii), we can conclude that that

$$u_{x_k} \in H^1(V) \quad \forall\, k \in \{1, 2, \ldots, n-1\},$$

with the estimate

$$\sum_{\substack{k,l=1 \\ k+l<2n}}^{n} \|u_{x_k x_l}\|_{L^2(V)} \leq C_2(\|f\|_{L^2(U)} + \|u\|_{H^1(U)}).$$

In this case, the restatement is because we don't have $V \subset\subset U$.

Proof of Theorem 6.3.4

Recall that in Step 7 of the proof of Theorem 6.3.1 (along with a homework problem), we saw that we can replace $\|u\|_{H^1(U)}$ on the right-hand side of this last inequality with $\|u\|_{L^2(U)}$. This gets us to

$$\sum_{\substack{k,l=1 \\ k+l<2n}}^{n} \|u_{x_k x_l}\|_{L^2(V)} \leq C_3(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}).$$

5. We still need an estimate on $\|u_{x_n x_n}\|_{L^2(V)}$. For this recall from a note following our statement of Theorem 6.3.1 that interior regularity allows us to work with the strong form of our equation, $Lu = f$ for a.e. $\vec{x} \in U$. To take advantage of this, let's first write our equation in the non-divergence form

$$-\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} \tilde{b}^i u_{x_i} + cu = f, \quad \tilde{b}^i := b^i - \sum_{j=1}^{n} a^{ij}_{x_j}.$$

Proof of Theorem 6.3.4

We can now isolate $u_{x_n x_n}$ as

$$a^{nn} u_{x_n x_n} = - \sum_{\substack{i,j=1 \\ i+j<2n}}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} \tilde{b}^i u_{x_i} + cu - f.$$

Our uniform ellipticity condition is

$$\sum_{i,j=1}^{n} a^{ij}(\vec{x}) \xi_i \xi_j \geq \theta |\vec{\xi}|^2, \quad \forall \vec{\xi} \in \mathbb{R}^n,$$

for some $\theta > 0$ and all $\vec{x} \in U$. In particular, if we take $\vec{\xi} = \hat{e}_n$, we see that

$$a^{nn}(\vec{x}) \geq \theta, \quad \forall \vec{x} \in U.$$

Proof of Theorem 6.3.4

This allows us to divide our relation for $u_{x_n x_n}$ by $a^{nn}(\vec{x})$ to obtain an inequality

$$|u_{x_n x_n}| \leq C_4 \Big(\sum_{\substack{i,j=1 \\ i+j<2n}}^{n} |u_{x_i x_j}| + |Du| + |u| + |f| \Big),$$

for a.e. $\vec{x} \in U$. If we square this inequality and integrate both sides over V, we obtain

$$\|u_{x_n x_n}\|_{L^2(V)} \leq C_5 \Big(\sum_{\substack{i,j=1 \\ i+j<2n}}^{n} \|u_{x_i x_j}\|_{L^2(V)} + \|Du\|_{L^2(V)} + \|u\|_{L^2(V)} + \|f\|_{L^2(V)} \Big).$$

Combining this inequality with our previous observations, we see that

$$\|u\|_{H^2(V)} \leq C_6 \Big(\|u\|_{L^2(U)} + \|f\|_{L^2(U)} \Big),$$

for some constant C_6.

Proof of Theorem 6.3.4

6. In the case of a general C^2 boundary, we fix any $\vec{x}_0 \in \partial U$, and we let $\vec{\Phi}(\vec{x})$ denote our usual straightening map, noting that for some $r > 0$ sufficiently small $\vec{\Phi}$ is a C^2 function on $B^o(\vec{x}_0, r)$ with C^2 inverse $\vec{\Psi}(\vec{y})$. We'll label the range of $\vec{\Phi}$ so that $\vec{\Phi}(\vec{x}_0) = 0$. (I.e., $\vec{y}_0 = 0$.)

7. We choose $s > 0$ sufficiently small so that

$$U' := B^o(0, s) \cap \{y_n > 0\} \subset \vec{\Phi}(U \cap B(\vec{x}_0, r)),$$

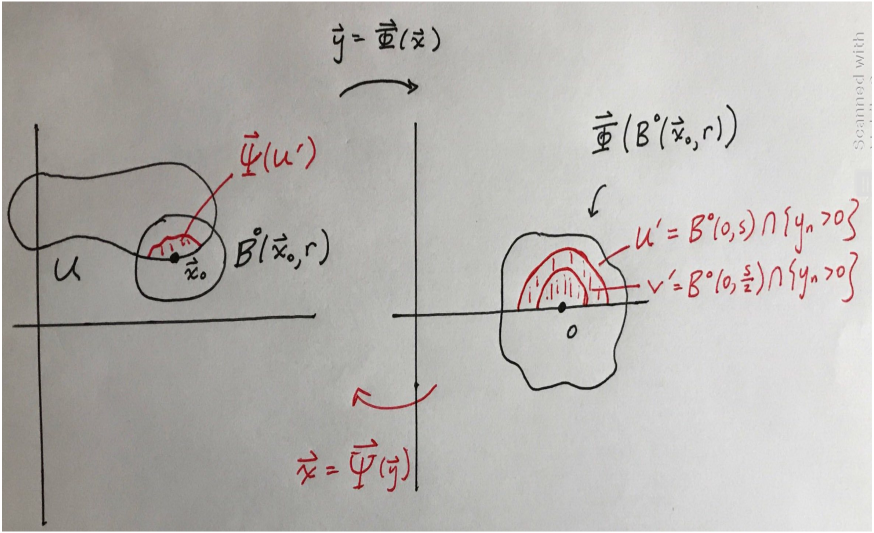
and correspondingly we set

$$V' := B^o(0, \frac{s}{2}) \cap \{y_n > 0\}.$$

We also set

$$u'(\vec{y}) := u(\Psi(\vec{y})), \quad \vec{y} \in U'.$$

See figure on the next slide.


$\vec{y} = \vec{\Phi}(\vec{x})$
$\vec{\Psi}(U')$
$B^\circ(\vec{x}_0, r)$
U
$\vec{x}_0$
$\vec{\Phi}(B^\circ(\vec{x}_0, r))$
$U' = B^\circ(0,s) \cap \{y_n > 0\}$
$V' = B^\circ(0,\frac{s}{2}) \cap \{y_n > 0\}$
0
$\vec{x} = \vec{\Psi}(\vec{y})$
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Proof of Theorem 6.3.4

We'll check the following claims in the homework:

1. $u' \in H^1(U')$
2. $u' = 0$ on $\partial U' \cap \{y_n = 0\}$ in the trace sense

8. In the new variables, our elliptic PDE can be expressed as

$$L'u' = f' \quad \text{in } U',$$

where

$$f'(\vec{y}) := f(\vec{\Psi}(\vec{y})),$$

and

$$L'u' := -\sum_{k,l=1}^{n} (a'^{kl} u'_{y_k})_{y_l} + \sum_{k=1}^{n} b'^{k} u'_{y_k} + c'u'.$$

The coefficients a'^{kl}, b'^{k}, and c' are given below.

Proof of Theorem 6.3.4

The coefficients a'^{kl}, b'^{k}, and c' are obtained directly by expressing the original operator L in the variable $\vec{y} = \vec{\Phi}(\vec{x})$ ($\iff$ $\vec{x} = \vec{\Psi}(\vec{y})$). Clearly,

$$c'(\vec{y}) = c(\vec{\Psi}(\vec{y})).$$

For the first-order term (in original variables)

$$\sum_{r=1}^{n} b^r(\vec{x}) u_{x_r}(\vec{x}),$$

we need to carry out a short calculation, and we'll do that on the next slide. For this, we'll denote by Φ^k the k^{th} component of $\vec{\Phi}$.

Proof of Theorem 6.3.4

We have

$$\begin{aligned}
\sum_{r=1}^{n} b^r(\vec{\Psi}(\vec{y}))\frac{\partial}{\partial x_r}u(\vec{\Psi}(\vec{y})) &= \sum_{r=1}^{n} b^r(\vec{\Psi}(\vec{y}))\frac{\partial}{\partial x_r}u'(\vec{y}) \\
&= \sum_{r=1}^{n} b^r(\vec{\Psi}(\vec{y}))D_y u'(\vec{y})\frac{\partial}{\partial x_r}\vec{\Phi}(\vec{x}) \\
&= \sum_{r=1}^{n} b^r(\vec{\Psi}(\vec{y}))\sum_{k=1}^{n}\Phi^k_{x_r}(\vec{\Psi}(\vec{y}))u'_{y_k}(\vec{y}) \\
&= \sum_{k=1}^{n}\left\{\sum_{r=1}^{n} b^r(\vec{\Psi}(\vec{y}))\Phi^k_{x_r}(\vec{\Psi}(\vec{y}))\right\}u'_{y_k}(\vec{y}).
\end{aligned}$$

We see that

$$b'^{k}(\vec{y}) = \sum_{r=1}^{n} b^r(\vec{\Psi}(\vec{y}))\Phi^k_{x_r}(\vec{\Psi}(\vec{y})).$$

Proof of Theorem 6.3.4

Proceeding similarly for the second-order term, we find that

$$a'^{kl}(\vec{y}) = \sum_{r,s=1}^{n} a^{rs}(\vec{\Psi}(\vec{y}))\Phi^k_{x_r}(\vec{\Psi}(\vec{y}))\Phi^l_{x_s}(\vec{\Psi}(\vec{y})).$$

Claim. $u'(\vec{y}) = u(\vec{\Psi}(\vec{y}))$ is a weak solution of

$$L'u' = f' \quad \text{in } U'.$$

In order to see this, we take any $v' \in H^1_0(U')$ and let $B'[u', v']$ denote the bilinear form associated with L',

$$B'[u', v'] = \int_{U'} \left\{ \sum_{k,l=1}^{n} a'^{kl} u'_{y_k} v'_{y_l} + \sum_{k=1}^{n} b'^{k} u'_{y_k} v' + c' u' v' \right\} d\vec{x}.$$

Proof of Theorem 6.3.4

Also, we set

$$v(\vec{x}) = v'(\vec{\Phi}(\vec{x})),$$

and observe that similarly as in the homework problem above, $v \in H_0^1(\vec{\Psi}(U'))$. In addition, it will be convenient below to extend v as 0 on $U \backslash \vec{\Psi}(U')$.

This will allow us to express $B'[u', v']$ in terms of u and v, which is what we do next.

Proof of Theorem 6.3.4

For the first-order term in $B'[u', v']$, we can write

$$\sum_{k=1}^{n} b'^{k} u'_{y_k} v' = \sum_{k=1}^{n} b'^{k} \Big(\frac{\partial}{\partial y_k} u(\vec{\Psi}(\vec{y}))\Big) v(\vec{\Psi}(\vec{y}))$$
$$= \sum_{k=1}^{n} b'^{k} (Du)(\vec{\Psi}(\vec{y})) \vec{\Psi}_{y_k}(\vec{y}) v(\vec{\Psi}(\vec{y}))$$
$$= \sum_{k=1}^{n} b'^{k} \sum_{i=1}^{n} u_{x_i}(\vec{\Psi}(\vec{y})) \Psi^{i}_{y_k}(\vec{y}) v(\vec{\Psi}(\vec{y})).$$

For notational brevity, we'll write this last expression as

$$\sum_{i=1}^{n} \sum_{k=1}^{n} b'^{k} u_{x_i} \Psi^{i}_{y_k} v.$$

Proceeding similarly for the other two terms, we obtain the relationship on the next slide.

Proof of Theorem 6.3.4

We have

$$B'[u', v'] = \sum_{i,j=1}^{n} \sum_{k,l=1}^{n} \int_{U'} a'^{kl} u_{x_i} \Psi^i_{y_k} v_{x_j} \Psi^j_{y_l} d\vec{y}$$

$$+ \sum_{i=1}^{n} \sum_{k=1}^{n} \int_{U'} b'^{k} u_{x_i} \Psi^i_{y_k} v d\vec{y} + \int_{U'} c' uv d\vec{y}.$$

According to our definition of a'^{kl}, we can write

$$\sum_{k,l=1}^{n} a'^{kl} \Psi^i_{y_k} \Psi^j_{y_l} = \sum_{k,l=1}^{n} \sum_{r,s=1}^{n} a^{rs} \Phi^k_{x_r} \Phi^l_{x_s} \Psi^i_{y_k} \Psi^j_{y_l}$$

$$= \sum_{k,l=1}^{n} \{(D\Phi^k) A (D\Phi^l)^T\} \Psi^i_{y_k} \Psi^j_{y_l}$$

$$= (D\Psi^i)\{(D\vec{\Phi}) A (D\vec{\Phi})^T\}(D\Psi^j)^T.$$

Proof of Theorem 6.3.4

Recall from our construction of the maps $\vec{\Phi}$ and $\vec{\Psi}$ last semester that $D\vec{\Psi} = (D\vec{\Phi})^{-1}$ (just differentiate the relation $\vec{\Psi}(\vec{\Phi}(\vec{x})) = \vec{x}$). We see that

$$(D\Psi^i)D\vec{\Phi} = \hat{e}_i^T \quad \forall\, i \in \{1, 2, \ldots, n\}.$$

In this way, we see that

$$\sum_{k,l=1}^{n} a'^{\,kl} \Psi^i_{y_k} \Psi^j_{y_l} = \hat{e}_i^T A \hat{e}_j = a^{ij}.$$

Similarly,

$$\sum_{k=1}^{b} b'^{\,k} \Psi^i_{y_k} = \sum_{k=1}^{n} \sum_{r=1}^{n} b^r \Phi^k_{x_r} \Psi^i_{y_k} = \sum_{r=1}^{n} b^r \sum_{k=1}^{n} (D\vec{\Psi})_{ik} (D\vec{\Phi})_{kr}$$

$$= \sum_{r=1}^{n} b^r \{(D\vec{\Psi})(D\vec{\Phi})\}_{ir} = b^i.$$

Proof of Theorem 6.3.4

Combining these observations, and using the change of variables $\vec{y} = \vec{\Phi}(\vec{x})$ (recall that $\det D\vec{\Phi}(\vec{x}) = 1$)

$$
\begin{aligned}
B'[u', v'] &= \sum_{i,j=1}^{n} \sum_{k,l=1}^{n} \int_{U'} a'^{kl} u_{x_i} \Psi^i_{y_k} v_{x_j} \Psi^j_{y_l} d\vec{y} \\
&\quad + \sum_{i=1}^{n} \sum_{k=1}^{n} \int_{U'} b'^{k} u_{x_i} \Psi^i_{y_k} v d\vec{y} + \int_{U'} c' uv d\vec{y} \\
&\overset{\vec{y}=\vec{\Phi}(\vec{x})}{=} \int_U \Big\{ \sum_{i,j=1}^{n} \Big[\sum_{k,l=1}^{n} a'^{kl} \Psi^i_{y_k} \Psi^j_{y_l} \Big] u_{x_i} v_{x_j} \\
&\quad + \sum_{i=1}^{n} \Big[\sum_{k=1}^{n} b'^{k} \Psi^i_{y_k} \Big] u_{x_i} v + cuv \Big\} d\vec{x} \\
&= \int_U \Big\{ \sum_{i,j=1}^{n} a^{ij} u_{x_i} v_{x_j} + \sum_{i=1}^{n} b^i u_{x_i} v + cuv \Big\} d\vec{x} = B[u, v].
\end{aligned}
$$

Proof of Theorem 6.3.4

The latter integrals can be expressed over U because v is taken to be 0 on $U \backslash \Psi(U')$. We see that for all $v' \in H_0^1(U')$,

$$B'[u', v'] = B[u, v] = (f, v) = (f', v'),$$

and so u' is a weak solution of $L'u' = f'$.

9. We will proceed by applying Steps 1-5 to L', and for this, we need to verify that L' satisfies our assumptions on L in the theorem.

Recalling the relation

$$a'^{kl}(\vec{y}) = \sum_{r,s=1}^{n} a^{rs}(\vec{\Psi}(\vec{y}))\Phi^k_{x_r}(\vec{\Psi}(\vec{y}))\Phi^l_{x_s}(\vec{\Psi}(\vec{y})),$$

we see that our assumption of a C^2 boundary ensures us that $a'^{kl} \in C^1(\bar{U}')$ for all $k, l \in \{1, 2, \ldots, n\}$. It's clear that $b'^{k}, c \in L^\infty(U')$, for all $k \in \{1, 2, \ldots, n\}$.

Proof of Theorem 6.3.4

We also need to check that L' is uniformly elliptic in U'. For this, we take any $\vec{y} \in U'$ and any $\vec{\xi} \in \mathbb{R}^n$ and we compute

$$\sum_{k,l=1}^{n} a'^{kl}(\vec{y})\xi_k \xi_l = \sum_{k,l=1}^{n} \sum_{r,s=1}^{n} a^{rs}(\vec{\Psi}(\vec{y}))\Phi^k_{x_r}(\vec{\Psi}(\vec{y}))\Phi^l_{x_s}(\vec{\Psi}(\vec{y}))\xi_k \xi_l$$
$$= \sum_{r,s=1}^{n} a^{rs}(\vec{\Psi}(\vec{y}))\eta_r \eta_s \geq \theta|\vec{\eta}|^2,$$

where we've set (with $\vec{\xi}$ viewed as a row vector)

$$\vec{\eta}(\vec{y}) := \vec{\xi} D\vec{\Phi}(\vec{\Psi}(\vec{y})) \implies \vec{\xi} = \vec{\eta}(\vec{y})(D\vec{\Phi})^{-1} = \vec{\eta}(\vec{y})D\vec{\Psi}(\vec{y}).$$

We see that

$$|\vec{\xi}| \leq C_7 |\vec{\eta}(\vec{y})|,$$

for some constant C_7.

Proof of Theorem 6.3.4

In particular,

$$\theta|\vec{\eta}|^2 \geq \frac{\theta}{C_7^2}|\vec{\xi}|^2,$$

and this gives uniform ellipticity with constant θ/C_7^2.

10. We are now justified in applying Steps 1-5 to u' as a solution of $L'u' = f'$, and this provides the inequality

$$\|u'\|_{H^2(V')} \leq C_8(\|f'\|_{L^2(U')} + \|u'\|_{L^2(U')}).$$

Returning to original variables, we can express this as

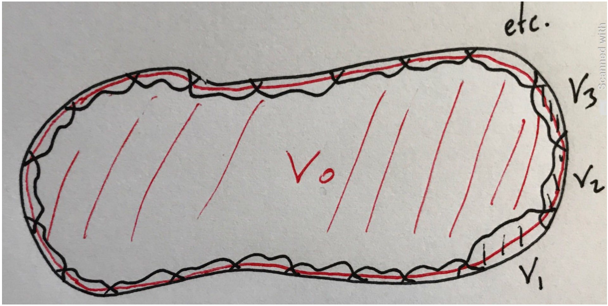
$$\|u\|_{H^2(V)} \leq C_9(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}),$$

where $V = \vec{\Psi}(V')$ and on the right-hand side we've extended the domain of integration from $\vec{\Psi}(U')$ to U.

Proof of Theorem 6.3.4

We now put these local estimates together in the usual way. Since ∂U is compact, we can find finitely many sets $\{V_i\}_{i=1}^N$ as described above, along with one additional set $V_0 \subset\subset U$ so that

$$U = \bigcup_{i=0}^{N} V_i.$$



Proof of Theorem 6.3.4

From Steps 1-9 of the current proof, we have

$$\|u\|_{H^2(V_i)} \leq K_i(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}), \quad i = 1, 2, \dots N,$$

for some constants $\{K_i\}_{i=1}^N$, while from Theorem 6.3.1 we have

$$\|u\|_{H^2(V_0)} \leq K_0(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}),$$

for some constant K_0. Finally,

$$\|u\|^2_{H^2(U)} \leq \sum_{i=0}^{N} \|u\|^2_{H^2(V_i)}$$

(inequality because of overlap in the sets), so for some constant C we have the claimed inequality

$$\|u\|_{H^2(U)} \leq C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}). \qquad \square$$

Higher Boundary Regularity

We'll conclude by summarizing two results on higher boundary regularity.

Theorem 6.3.5. Suppose $U \subset \mathbb{R}^n$ is open and bounded with C^{m+2} boundary, a^{ij}, b^i, $c \in C^{m+1}(\bar{U})$ $(\forall\, i,j \in \{1,2,\dots n\})$ for some $m \in \{0,1,2,\dots\}$, and L is uniformly elliptic. Also, suppose $f \in H^m(U)$, and $u \in H_0^1(U)$ is a weak solution of

$$Lu = f \quad \text{in } U$$
$$u = 0, \quad \text{on } \partial U.$$

Then $u \in H^{m+2}(U)$, and there exists a constant C, depending only on m, U and the coefficients of L, so that

$$\|u\|_{H^{m+2}(U)} \leq C(\|f\|_{H^m(U)} + \|u\|_{L^2(U)}).$$

Note. If Reg $(H^{m+2}) > 0$, then u is continuous up to the boundary.

Higher Boundary Regularity

Theorem 6.3.6. Suppose $U \subset \mathbb{R}^n$ is open and bounded with C^∞ boundary, a^{ij}, b^i, $c \in C^\infty(\bar{U})$ $(\forall\, i, j \in \{1, 2, \ldots n\})$, and L is uniformly elliptic. Also, suppose $f \in C^\infty(\bar{U})$, and $u \in H^1_0(U)$ is a weak solution of

$$\begin{aligned} Lu &= f \quad \text{in } U \\ u &= 0, \qquad \text{on } \partial U. \end{aligned}$$

Then $u \in C^\infty(\bar{U})$.