Second Order Parabolic PDE: Background on Function Spaces Involving Time, Part I

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Overview

When we analyze evolutionary PDE such as the heat equation

$$u_t = \Delta u,$$

it will useful to view the equation as an ODE with dependent variable taking values in an appropriate Banach space X. I.e., for some value $T > 0$, we will think of $u = u(t)$ as a map $u : [0, T] \to X$.

For some background on such functions, we'll briefly return to Chapter 5 in Evans, Section 5.9.2 on Spaces involving time.

Continuity and Differentiability

Continuity. We say that a function $u : [0, T] \to X$ is continuous at a point $t_0 \in (0, T)$ if given any $\epsilon > 0$ there exists some $\delta > 0$ so that

$$|t - t_0| < \delta \implies \|u(t) - u(t_0)\|_X < \epsilon.$$

As usual, we can define continuity at endpoints in a one-sided fashion.

Differentiability. We say that a function $u : [0, T] \to X$ is differentiable at a point $t_0 \in (0, T)$ if there exists some $u'(t_0) \in X$ and corresponding map $\epsilon(h; t_0)$, so that

$$u(t_0 + h) = u(t_0) + u'(t_0)h + \epsilon(h; t_0),$$

where

$$\lim_{h \to 0} \frac{\|\epsilon(h; t_0)\|_X}{h} = 0.$$

Easy Examples

1. If $u_0 \in X$ is any fixed element, and $\zeta \in C([0,T];\mathbb{R})$, then

$$u(t) = \zeta(t) u_0$$

is a continuous map $u : [0,T] \to X$. Likewise, if ζ is differentiable at some $t_0 \in (0,T)$ then so is $u(t)$, and $u'(t_0) = \zeta'(t_0) u_0$.

2. If $v : [0,T] \to X$ is continuous on $[0,T]$ and $\zeta \in C([0,T];\mathbb{R})$, then

$$u(t) = \zeta(t) v(t)$$

is a continuous map $u : [0,T] \to X$. Likewise, if v and ζ are both differentiable at some $t_0 \in (0,T)$ then so is $u(t)$, and

$$u'(t_0) = \zeta'(t_0) v(t_0) + \zeta(t_0) v'(t_0).$$

The Spaces $C^k(\bar{I}; X)$

Definition. For an open interval $I \subset \mathbb{R}$, we denote by $C(\bar{I}; X)$ the collection of all uniformly continuous functions $u : I \to X$, for which we have

$$\|u\|_{C(\bar{I};X)} := \sup_{0<t<T} \|u(t)\|_X < \infty.$$

If $I = (0, T)$, then $C(\bar{I}; X)$ is equivalent to the set $C([0, T]; X)$ that Evans defines on p. 301.

We denote by $C^k(\bar{I}; X)$ the collection of all uniformly continuous functions $u : I \to X$ for which the first k derivatives of u are also uniformly continous as maps $I \mapsto X$, and for which we have

$$\|u\|_{C^k(\bar{I};X)} := \sum_{j=0}^{k} \|u^{(j)}\|_{C(\bar{I};X)} < \infty.$$

We denote by $C^\infty(\bar{I}; X)$ the collection of all $u : I \to X$ so that $u \in C^k(\bar{I}; X)$ for all $k = 0, 1, 2, \ldots$.

The Spaces $C_c^k(I; X)$

Theorem 1. If X is a Banach space, then for each $k \in \{0, 1, 2, \dots\}$, the space $C^k(\bar{I}; X)$ is a Banach space.

Definition. Let $I \subset \mathbb{R}$ be an open interval. For each $k \in \{0, 1, 2, \dots\}$, we denote by $C_c^k(I; X)$ the space of all maps $u : I \to X$ so that u and its first k derivatives are continuous on I, and spt $(u) \subset [a, b] \subset I$.

We denote by $C_c^\infty(I; X)$ the collection of all maps $u : I \to X$ so that u and its derivatives to every order are continuous on I, and spt $(u) \subset [a, b] \subset I$.

Riemann Integration

Definition. For a map $u : [0, T] \to X$, we can define the usual Riemann sums

$$\mathcal{R}_P = \sum_{i=1}^{N} u(\bar{t}_i)(t_i - t_{i-1}),$$

where P denotes a partition of $[0, T]$,
$0 = t_0 < t_1 < t_2 < \cdots < t_N = T$, and for each $i \in \{1, 2, \ldots, N\}$,
$\bar{t}_i \in [t_{i-1}, t_i]$. We denote the mesh of the partition as
$\Delta P := \max_{i \in \{1,2,\ldots,N\}} |t_i - t_{i-1}|$.

If $\lim_{k\to\infty} \mathcal{R}_{P_k}$ exists for all sequences of partitions $\{P_k\}_{k=1}^{\infty}$ so that $\Delta P_k \to 0$ as $k \to \infty$, then we define the mutual limit as the Riemann integral of u

$$\int_0^T u(t)dt := \lim_{k\to\infty} \mathcal{R}_{P_k}.$$

Riemann Integration

Theorem 2. If X is a Banach space and $u \in C([0,T];X)$, then the following hold:

(i) For all $t \in [0,T]$, the Riemann integral $\int_0^t u(s)ds$ exists.

(ii) The Riemann integral $\int_0^t u(s)ds$ is differentiable at each $t \in (0,T)$, and

$$\frac{d}{dt}\int_0^t u(s)ds = u(t).$$

(iii) If $u' \in C((0,T);X)$, then for any $0 < s < t < T$,

$$u(t) = u(s) + \int_s^t u'(\tau)d\tau.$$

Bochner Integration

The usual development of Lebesgue integration as the supremum of integrals of simple functions relies on the ordering properties of $\mathbb{R}$, and can't be directly extended to the current setting.

Instead, we proceed by completing the space of simple functions with respect to a norm that we will denote $L^1((0, T); X)$ below. This leads to the Bochner integral of a function (named after the US mathematician, Salomon Bochner (1899)-(1982)).

Bochner Integration

Definitions.

(i) A function $s : [0, T] \to X$ is called a simple function if it has the form

$$s(t) = \sum_{i=1}^{m} \chi_{E_i}(t) u_i,$$

where each E_i is a Lebesque measurable subset of $[0, T]$, χ_{E_i} denotes a characteristic function on E_i, and $u_i \in X$ for each $i \in \{1, 2, \dots, m\}$. In this case, we define

$$\int_0^T s(t)dt := \sum_{i=1}^{m} |E_i| u_i \in X.$$

(ii) A function $u : [0, T] \to X$ is called strongly measurable if there exist simple functions $\{s_k\}_{k=1}^{\infty} : [0, T] \to X$ so that

$$s_k(t) \to u(t) \quad \text{in } X \text{ for a.e. } 0 < t < T.$$

Bochner Integration

(iii) We say that a strongly measurable function $u : [0, T] \to X$ is (Bochner) integrable if there exists a sequence of simple functions $\{s_k\}_{k=1}^{\infty} : [0, T] \to X$ so that

$$\lim_{k\to\infty} \int_0^T \|s_k(t) - u(t)\|_X \, dt = 0. \qquad (*)$$

In this case, we define

$$\int_0^T u(t)dt := \lim_{k\to\infty} \int_0^T s_k(t)dt \in X.$$

Here, using the form of simple functions and (*), we see that the sequence $\{\int_0^T s_k(t)dt\}_{k=1}^{\infty}$ is Cauchy in X, and so converges in X.

Theorem A.E.8. A strongly measurable function $u : [0, T] \to X$ is integrable if and only if the map $t \mapsto \|u(t)\|_X$ is integrable. In this case,

$$\| \int_0^T u(t)dt\|_X \leq \int_0^T \|u(t)\|_X dt,$$

and

$$\langle v^*, \int_0^T u(t)dt\rangle = \int_0^T \langle v^*, u(t)\rangle dt$$

for all $v^* \in X^*$.

Bochner Integration

Note. As an important special case of the latter claim, suppose X and Y are two Banach spaces, and X is continuously embedded in Y. Recalling that this means that the identity map $I : X \to Y$ is a bounded linear operator, we see that

$$I \int_0^T u(t)dt = \int_0^T Iu(t)dt.$$

I.e., the integrals associated with $u : [0, T] \to X$ and $Iu : [0, T] \to Y$ agree.

The Spaces $L^p(0, T; X)$

Definition. We denote by $L^p(0, T; X)$ the space of all strongly measurable functions $u : [0, T] \to X$ so that

$$\|u\|_{L^p(0,T;X)} := \left(\int_0^T \|u(t)\|_X^p dt \right)^{1/p} < \infty \quad (1 \leq p < \infty)$$

$$\|u\|_{L^p(0,T;X)} := \operatorname*{ess\,sup}_{0 \leq t \leq T} \|u(t)\|_X < \infty \quad (p = \infty).$$

For $1 \leq p \leq \infty$, we'll denote by $L^p_{\mathrm{loc}}(0, T; X)$ the space of strongly measurable functions $u : [0, T] \to X$ so that $u \in L^p(a, b; X)$ for all $[a, b] \subset (0, T)$.

In the usual way, if $u(t) = v(t)$ in X for a.e. $t \in (0, T)$, we equate u and v in $L^p(0, T; X)$.

Theorem 3. For any $1 \leq p \leq \infty$, if X is a Banach space, then $L^p(0, T; X)$ is a Banach space.

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Theorem 4. Suppose X is a Banach space, and a sequence of functions $\{f_k(t)\}_{k=1}^{\infty} \subset L^1(0, T; X)$ satisfies

$$f_k(t) \to f(t) \quad \text{in } X$$

for a.e. $t \in (0, T)$. Suppose also that there exists $g \in L^1(0, T; X)$ so that

$$\|f_k(t)\|_X \leq \|g(t)\|_X$$

for a.e. $t \in (0, T)$. Then $f \in L^1(0, T; X)$ and

$$\lim_{k\to\infty} \int_0^T f_k(t)dt = \int_0^T f(t)dt \quad \text{in } X,$$

with also

$$\lim_{k\to\infty} \int_0^T \|f_k(t) - f(t)\|_X dt = 0.$$

Approximation

Theorem 5. Suppose X is a Banach space and $1 \le p < \infty$. Then the set $C_c^\infty((0,T),X)$ is dense in $L^p(0,T;X)$. In fact, more is true. The collection of functions of the form

$$u(t) = \sum_{i=1}^{N} u_i \zeta_i(t); \quad \zeta_i \in C_c^\infty(0,T;\mathbb{R}), \quad u_i \in X$$

is dense in $L^p(0,T;X)$.

Here, N isn't fixed, but rather indicates that the sums are finite.

Mollification in $L^1_{\mathrm{loc}}(0, T; X)$

Let

$$\eta(t) = \begin{cases} Ce^{\frac{1}{t^2-1}} & |t| < 1 \\ 0 & |t| \geq 1, \end{cases}$$

where

$$C = \frac{1}{\int_{-1}^{1} e^{\frac{1}{t^2-1}}\, dt} \Rightarrow \int_{-\infty}^{+\infty} \eta(t)dt = 1.$$

Then $\eta \in C^\infty(\mathbb{R})$ and spt $(\eta) \subset [-1, 1]$. As usual, we set

$$\eta_\epsilon(t) := \frac{1}{\epsilon}\eta(\frac{t}{\epsilon}) \Rightarrow \text{spt } \eta_\epsilon \subset (-\epsilon, \epsilon),$$

and also

$$f^\epsilon(t) := \eta_\epsilon * f(t) = \int_{-\infty}^{+\infty} \eta_\epsilon(t-\tau)f(\tau)d\tau$$

$$= \int_{t-\epsilon}^{t+\epsilon} \eta_\epsilon(t-\tau)f(\tau)d\tau.$$

Mollification in $L^1_{\mathrm{loc}}(0, T; X)$

Theorem 6. Let X denote a Banach space and suppose $f \in L^1_{\mathrm{loc}}(0, T; X)$ for some $T > 0$. For

$$f^\epsilon(t) := \eta_\epsilon * f(t) \quad \text{in } (\epsilon, T - \epsilon)$$

we have the following:

(i) $f^\epsilon \in C^\infty(\epsilon, T - \epsilon; X)$, and

$$\partial_t^k f^\epsilon(t) = (\partial_t^k \eta_\epsilon) * f(t); \quad k = 1, 2, \ldots .$$

(ii) $f^\epsilon(t) \to f(t)$ in X as $\epsilon \to 0$ for a.e. $t \in (0, T)$.

(iii) If $f \in C((0, T); X)$ then $f^\epsilon \to f$ uniformly in X as $\epsilon \to 0$ on compact subsets of $(0, T)$.

(iv) If $1 \leq p < \infty$ and $f \in L^p_{\mathrm{loc}}((0, T); X)$, then $f^\epsilon \to f$ in $L^p_{\mathrm{loc}}((0, T); X)$ as $\epsilon \to 0$.