Second Order Parabolic PDE: Background on Function Spaces Involving Time, Part II

MATH 612, Texas A&M University

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Weak Derivatives

Definition. We say that a function $u \in L^1(0, T; X)$ is weakly differentiable on $(0, T)$ if there exists a function $v \in L^1(0, T; X)$ so that

$$\int_0^T u(t)\phi'(t)dt = -\int_0^T v(t)\phi(t)dt$$

for all $\phi \in C_c^\infty(0, T; \mathbb{R})$. We say that v is the weak derivative of u and write $u' = v$.

Notes. 1. The ϕ are our usual test functions, taking values in $\mathbb{R}$.

2. We're following the convention Evans adopts and using $u \in L^1(0, T; X)$, but we could also use $u \in L^1_{\mathrm{loc}}(0, T; X)$.

Weak Derivatives

3. Suppose X and Y are Banach spaces with X continuously embedded in Y, and denote by $I : X \to Y$ the identity map. In addition, suppose $u \in L^1(0, T; X)$ and $(Iu)' \in L^1(0, T; Y)$. Then according to the note following Theorem A.E.8, we can write

$$I\Big(\int_0^T u(t)\phi'(t)dt\Big) = \int_0^T (Iu(t))\phi'(t)dt = -\int_0^T (Iu)'(t)\phi(t)dt.$$

If we identify u with Iu, we can view $(Iu)'$ as the Y-valued derivative of the X-valued function u.

The Sobolev Space $W^{1,p}(0,T;X)$

Definition. We denote by $W^{1,p}(0,T;X)$ the space of all functions $u \in L^p(0,T;X)$ so that u' exists in the weak sense and belongs to $L^p(0,T;X)$. We equip $W^{1,p}(0,T;X)$ with the norms

$$\|u\|_{W^{1,p}(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p + \|u'(t)\|_X^p dt \right)^{1/p} \quad (1 \le p < \infty)$$

$$\|u\|_{W^{1,p}(0,T;X)} = \operatorname*{ess\,sup}_{0 \le t \le T} \left(\|u(t)\|_X + \|u'(t)\|_X \right) \quad (p = \infty).$$

We will use the notation $H^1(0,T;X) = W^{1,2}(0,T;X)$.

Theorem 7. For any $1 \le p \le \infty$, if X is a Banach space, then $W^{1,p}(0,T;X)$ is a Banach space.

Sobolev Embedding and Calculus

Theorem 5.9.2. Let $u \in W^{1,p}(0, T; X)$ for some $1 \leq p \leq \infty$. Then:

(i) $u \in C([0, T]; X)$ (for some version of u)

(ii) For all $0 \leq s \leq t \leq T$

$$u(t) = u(s) + \int_s^t u'(\tau) d\tau.$$

(iii) There exists a constant C, depending only on T, so that

$$\|u\|_{C([0,T];X)} \leq C\|u\|_{W^{1,p}(0,T;X)}.$$

Note. Here, for the variable t, we have $n = 1$, so formally, Reg $(W^{1,p}(0, T; X)) = 1 - \frac{1}{p} \geq 0$ for all $1 \leq p \leq \infty$.

Proof of Theorem 5.9.2

0. Two familiar observations will come up during this proof in a new setting. First, if $u \in L^p(0, T; X)$, then since $(0, T)$ is bounded, we can conclude that $u \in L^q(0, T; X)$ for all $1 \leq q \leq p$. Second, if $u \in L^1(0, T; X)$ then

$$\int_0^t u(s)ds$$

is continuous as a function of t.

1. We let $\eta_\epsilon(t)$ denote the usual real-valued mollifer, and set

$$u^\epsilon(t) = \eta_\epsilon * u(t) \quad \text{in } (\epsilon, T - \epsilon).$$

We know from Theorem 6 (i) that $u^\epsilon \in C^\infty(\epsilon, T - \epsilon; X)$, and

$$u^{\epsilon\prime}(t) = \eta_\epsilon' * u(t) \quad \text{in } (\epsilon, T - \epsilon).$$

Proof of Theorem 5.9.2

Claim. $u^{\epsilon\prime}(t) = \eta_\epsilon * u'(t)$ for all $t \in (\epsilon, T - \epsilon)$.

We did this calculation in the proof of Theorem 5.3.1, but there's no harm in doing it in this new setting as well. We write

$$\begin{aligned} u^{\epsilon\prime}(t) = \eta_\epsilon' * u(t) &= \int_{-\infty}^{+\infty} \eta_\epsilon'(t-\tau)u(\tau)d\tau \\ &= -\int_{-\infty}^{+\infty} \partial_\tau \eta_\epsilon(t-\tau)u(\tau)d\tau. \end{aligned}$$

For $t \in (\epsilon, T - \epsilon)$, $\eta_\epsilon(t - \tau)$ is 0 whenever u is undefined, but as Evans notes, we can extend u by 0 to $\mathbb{R}$ if we like.

Here, for each $t \in (\epsilon, T - \epsilon)$, $\eta_\epsilon(t - \cdot) \in C_c^\infty(0, T; \mathbb{R})$, so in particular $\eta_\epsilon(t - \tau)$ is a valid test function on $(0, T)$.

Proof of Theorem 5.9.2

Since u is weakly differentiable on $(0, T)$, we can conclude that for each $t \in (\epsilon, T-\epsilon)$,

$$-\int_{-\infty}^{+\infty} \partial_\tau \eta_\epsilon(t-\tau)u(\tau)d\tau = -\int_0^T \partial_\tau \eta_\epsilon(t-\tau)u(\tau)d\tau$$
$$= \int_0^T \eta_\epsilon(t-\tau)u'(\tau)d\tau = \int_{-\infty}^{+\infty} \eta_\epsilon(t-\tau)u'(\tau)d\tau,$$

and this is the claim.

According to Theorem 6 (ii), we know that as $\epsilon \to 0$

$$u^\epsilon(t) \to u(t) \quad \text{in } X \text{ for a.e. } t \in (0, T),$$

and, since ${u^\epsilon}'(t) = \eta_\epsilon * u'(t)$, with $u' \in L^p(0, T; X)$, we know that

$${u^\epsilon}' \to u' \quad \text{in } L^p_{\mathrm{loc}}(0, T; X).$$

Proof of Theorem 5.9.2

It follows from this latter convergence that

$$u^{\epsilon\,\prime} \to u' \quad \text{in } L^1_{\text{loc}}(0, T; X).$$

Notice that if $p = \infty$, then we don't have convergence in $L^\infty_{\text{loc}}(0, T; X)$, but we still have convergence in $L^1_{\text{loc}}(0, T; X)$.

According to Theorem 2, for any $0 < s < t < T$ we can write

$$u^\epsilon(t) = u^\epsilon(s) + \int_s^t u^{\epsilon\,\prime}(\tau) d\tau.$$

Using the pointwise convergence of $u^\epsilon(t)$ and the $L^1_{\text{loc}}(0, T; X)$ convergence of $u^{\epsilon\,\prime}$, we can conclude that for a.e. $0 < s < t < T$ we have

$$u(t) = u(s) + \int_s^t u'(\tau) d\tau.$$

Since the integral is continuous in both s and t, we see that in fact u is continuous on $[0, T]$. This gives both (i) and (ii).

Proof of Theorem 5.9.2

2. For Item (iii), we can use Theorem A.E.8 to write

$$\|u(t)\|_X \leq \left\| u(s) + \int_s^t u'(\tau)d\tau \right\|_X$$
$$\leq \|u(s)\|_X + \int_s^t \|u'(\tau)\|_X d\tau. \qquad (*)$$

We now integrate this relation with respect to s to see that

$$T\|u(t)\|_X \leq \int_0^T \|u(s)\|_X ds + \int_0^T \int_s^t \|u'(\tau)\|_X d\tau ds$$
$$\leq \int_0^T \|u(s)\|_X ds + T\int_0^T \|u'(\tau)\| d\tau.$$

Dividing by T, we see that there exists a constant $\tilde{C}$ so that

$$\|u(t)\|_X \leq \tilde{C}\int_0^T \|u(t)\|_X + \|u'(t)\|_X dt.$$

Proof of Theorem 5.9.2

The right-hand side does not depend on t, so we can take the maximum over $t \in [0, T]$ to see that

$$\|u\|_{C([0,T];X)} \leq \tilde{C} \int_0^T \|u(t)\|_X + \|u'(t)\|_X \, dt.$$

Since this integration is over a bounded domain, we can use Hölder's inequality in the usual way to get the claimed estimate. $\square$

Sobolev Embedding and More Calculus

Notes. 1. While studying second order parabolic PDE, we will often work with functions $u \in L^2(0, T; H_0^1(U))$ for which $u' \in L^2(0, T; H^{-1}(U))$. Intuitively, we can understand this by considering the heat equation

$$u_t = \Delta u.$$

In general, we expect the Laplacian to reduce the regularity of a function space by 2, so if $u \in H^m(U)$, then $\Delta u \in H^{m-2}(U)$.

Correspondingly, if $u : [0, T] \to H^m(U)$ is a solution of the heat equation, we expect $u' : [0, T] \to H^{m-2}(U)$. The case described above corresponds (very formally!) with $m = 1$.

Sobolev Embedding and More Calculus

2. In the setting of Note 1, we should keep in mind that we continue to have $u \in L^1(0, T; H_0^1(U))$ and $u' \in L^1(0, T; H_0^1(U))$, as specified in our definition of weak differentiability.

We recall that $H_0^1(U)$ is continuously embedded in $H^{-1}(U)$, and use the third note following our definition of weak derivatives to view the weak derivative of u as $(Iu)' \in L^1(0, T; H^{-1}(U))$.

When we write $u \in L^2(0, T; H_0^1(U))$, we mean that this is the case in addition to $u \in L^1(0, T; H_0^1(U))$, and likewise when we write $u' \in L^2(0, T; H^{-1}(U))$, we mean that in addition to $(Iu)' \in L^1(0, T; H^{-1}(U))$ we have $(Iu)' \in L^2(0, T; H^{-1}(U))$.

Sobolev Embedding and More Calculus

Theorem 5.9.3. Suppose $u \in L^2(0, T; H_0^1(U))$ and $u' \in L^2(0, T; H^{-1}(U))$. Then:

(i) $u \in C([0, T]; L^2(U))$ (for some version of u)

(ii) The mapping $t \mapsto \|u(t)\|_{L^2(U)}$ is absolutely continuous, and

$$\frac{d}{dt}\|u(t)\|^2_{L^2(U)} = 2\langle u'(t), u(t)\rangle,$$

for a.e. $0 < t < T$.

(iii) There exists a constant C, depending only on T, so that

$$\|u\|_{C([0,T];L^2(U))} \leq C\Big(\|u\|_{L^2(0,T;H_0^1(U))} + \|u'\|_{L^2(0,T;H^{-1}(U))}\Big).$$

Proof of Theorem 5.9.3

1. First, we extend u (by 0) to a slightly larger interval $[-\sigma, T+\sigma]$, $\sigma > 0$, so that we'll be able to evaluate mollifications of u on the full set $[0, T]$.

For $0 < \epsilon, \delta < \sigma$, we set $u^\epsilon = \eta_\epsilon * u$ and $u^\delta = \eta_\delta * u$. Then, for any $t \in (0, T)$

$$\begin{aligned}\frac{d}{dt}\|u^\epsilon(t) - u^\delta(t)\|^2_{L^2(U)} &= \frac{d}{dt}\int_U (u^\epsilon(t) - u^\delta(t))^2 d\vec{x} \\ &= 2\int_U (u^\epsilon(t) - u^\delta(t))(u^{\epsilon\prime}(t) - u^{\delta\prime}(t))d\vec{x} \\ &= 2(u^{\epsilon\prime}(t) - u^{\delta\prime}(t), u^\epsilon(t) - u^\delta(t))_{L^2(U)}.\end{aligned}$$

where we differentiate under the integral in the usual way with difference quotients and LDCT. Notice that in this last calculation, $u^{\epsilon\prime}(t) = (\eta'_\epsilon * u)(t) \in H^1_0(U)$.

Proof of Theorem 5.9.3

Integrating this relation, we obtain

$$\begin{aligned}\|u^\epsilon(t) - u^\delta(t)\|^2_{L^2(U)} &= \|u^\epsilon(s) - u^\delta(s)\|^2_{L^2(U)} \\ &+2\int_s^t (u^{\epsilon\prime}(\tau) - u^{\delta\prime}(\tau), u^\epsilon(\tau) - u^\delta(\tau))_{L^2(U)} d\tau, \qquad (*)\end{aligned}$$

for all $0 \le s, t \le T$. From Item (iii) of Theorem 5.9.1,

$$\begin{aligned}&(u^{\epsilon\prime}(\tau) - u^{\delta\prime}(\tau), u^\epsilon(\tau) - u^\delta(\tau))_{L^2(U)} \\ &\qquad = \langle u^{\epsilon\prime}(\tau) - u^{\delta\prime}(\tau), u^\epsilon(\tau) - u^\delta(\tau)\rangle,\end{aligned}$$

where for the right-hand side we mean the action of $u^{\epsilon\prime}(\tau) - u^{\delta\prime}(\tau)$, viewed as an element of $H^{-1}(U)$, on $u^\epsilon(\tau) - u^\delta(\tau) \in H_0^1(U)$.

If we compute the supremum of both sides of (*) over $t \in [0, T]$, we obtain the inequality on the next slide.

Proof of Theorem 5.9.3

We can now compute

$$\begin{aligned}\sup_{0\le t\le T}\|u^\epsilon(t)-u^\delta(t)\|^2_{L^2(U)} &\le \|u^\epsilon(s)-u^\delta(s)\|^2_{L^2(U)} \\ &+2\int_0^T \|u^{\epsilon\prime}(\tau)-u^{\delta\prime}(\tau)\|_{H^{-1}(U)}\|u^\epsilon(\tau)-u^\delta(\tau)\|_{H^1(U)}d\tau \\ &\le \|u^\epsilon(s)-u^\delta(s)\|^2_{L^2(U)}+\int_0^T \|u^{\epsilon\prime}(\tau)-u^{\delta\prime}(\tau)\|^2_{H^{-1}(U)}d\tau \\ &+\int_0^T \|u^\epsilon(\tau)-u(\tau)\|^2_{H^1(U)}d\tau \\ &\le \|u^\epsilon(s)-u^\delta(s)\|^2_{L^2(U)}+\|u^{\epsilon\prime}-u^{\delta\prime}\|^2_{L^2(0,T;H^{-1}(U))} \\ &+\|u^\epsilon(\tau)-u^\delta(\tau)\|^2_{L^2(0,T;H^1_0(U))}.\end{aligned}$$

Proof of Theorem 5.9.3

I.e., we have

$$\|u^\epsilon - u^\delta\|_{C([0,T];L^2(U))} \leq \|u^\epsilon(s) - u^\delta(s)\|^2_{L^2(U)} + \|u^{\epsilon\prime} - u^{\delta\prime}\|^2_{L^2(0,T;H^{-1}(U))} + \|u^\epsilon - u^\delta\|^2_{L^2(0,T;H^1_0(U))}.$$

Since $u \in L^2(0, T; H^1_0(U))$, we know from Theorem 6 (ii) that $u^\epsilon(t) \to u(t)$ in $H^1_0(U)$ for a.e. $t \in (0, T)$. We choose $s \in (0, T)$ to be one of these values.

We also know from Theorem 6 (iv) that as $\epsilon \to 0$

$$u^\epsilon \to u \quad \text{in } L^2_{\mathrm{loc}}(-\sigma, T+\sigma, H^1_0(U))$$
$$u^{\epsilon\prime} \to u' \quad \text{in } L^2_{\mathrm{loc}}(-\sigma, T+\sigma, H^{-1}(U)).$$

Combining these observations, we see that $\{u^\epsilon\}$ is Cauchy in $C([0, T]; L^2(U))$, and so there is some $v \in C([0, T]; L^2(U))$ so that $u^\epsilon \to v$ in $C([0, T]; L^2(U))$.

Proof of Theorem 5.9.3

Using again the observation that $u^\epsilon(t) \to u(t)$ in $H_0^1(U)$ for a.e. $t \in (0, T)$, we see that $u(t) = v(t)$ for a.e. $t \in (0, T)$. We can conclude that v is a continuous version of u, giving (i).

2. For (ii), we can use the same argument as in Step 1 to see that

$$\|u^\epsilon(t)\|^2_{L^2(U)} = \|u^\epsilon(s)\|^2_{L^2(U)} + 2\int_s^t \langle u^{\epsilon\prime}(\tau), u^\epsilon(\tau)\rangle d\tau,$$

for all $0 \leq s, t \leq T$. We can take the limit as $\epsilon \to 0$ in this expression, noting as above that $\|u^\epsilon(t)\|^2_{L^2(U)} \to \|u(t)\|_{L^2(U)}$ for a.e. $t \in (0, T)$.

Let's check that

$$\lim_{\epsilon \to 0} \int_s^t \langle u^{\epsilon\prime}(\tau), u^\epsilon(\tau)\rangle d\tau = \int_s^t \langle u'(\tau), u(\tau)\rangle d\tau.$$

Proof of Theorem 5.9.3

For this, we compute

$$
\begin{aligned}
&\Big| \int_s^t \langle u^{\epsilon\prime}(\tau), u^\epsilon(\tau)\rangle d\tau - \int_s^t \langle u'(\tau), u(\tau)\rangle d\tau \Big| \\
&\quad \leq \Big| \int_s^t \langle u^{\epsilon\prime}(\tau), u^\epsilon(\tau)\rangle - \langle u^{\epsilon\prime}(\tau), u(\tau)\rangle d\tau \Big| \\
&\qquad + \Big| \int_s^t \langle u^{\epsilon\prime}(\tau), u(\tau)\rangle - \langle u'(\tau), u(\tau)\rangle d\tau \Big| \\
&\quad \leq \int_0^T \|u^{\epsilon\prime}(\tau)\|_{H^{-1}(U)} \|u^\epsilon(\tau) - u(\tau)\|_{H^1(U)} d\tau \\
&\quad + \int_0^T \|u^{\epsilon\prime}(\tau) - u'(\tau)\|_{H^{-1}(U)} \|u(\tau)\|_{H^1(U)} d\tau.
\end{aligned}
$$

Proof of Theorem 5.9.3

We can now apply Hölder's inequality to each of these last two integrals. E.g., for the first, we have

$$\int_0^T \|u^{\epsilon\prime}(\tau)\|_{H^{-1}(U)} \|u^\epsilon(\tau) - u(\tau)\|_{H^1(U)} d\tau$$
$$\leq \Big(\int_0^T \|u^{\epsilon\prime}(\tau)\|^2_{H^{-1}(U)} \, d\tau\Big)^{1/2} \Big(\int_0^T \|u^\epsilon(\tau) - u(\tau)\|^2_{H^1(U)} d\tau\Big)^{1/2}$$
$$= \|u^{\epsilon\prime}\|_{L^2(0,T;H^{-1}(U))} \|u^\epsilon - u\|_{L^2(0,T;H^1(U))} \overset{\epsilon\to 0}{\to} 0.$$

The second is similar. We conclude that

$$\|u(t)\|^2_{L^2(U)} = \|u(s)\|^2_{L^2(U)} + 2\int_s^t \langle u'(\tau), u(\tau)\rangle d\tau, \qquad (**)$$

for a.e. $0 < s, t < T$, and if we take u to be its continuous version, we must have the relation for all $0 \leq s, t \leq T$. Since $\langle u'(\tau), u(\tau)\rangle \in L^1(0, T)$, we can conclude that $\|u(t)\|^2_{L^2(U)}$ is absolutely continuous. Upon differentiation of (**) we obtain (ii).

Proof of Theorem 5.9.3

3. For (iii) if we integrate our relation

$$\|u(t)\|^2_{L^2(U)} = \|u(s)\|^2_{L^2(U)} + 2\int_s^t \langle u'(\tau), u(\tau)\rangle d\tau,$$

in s on the interval $[0, T]$, we see that

$$\begin{aligned}
T\|u(t)\|^2_{L^2(U)} &= \int_0^T \|u(s)\|^2_{L^2(U)} ds + 2\int_0^T \int_s^t \langle u'(\tau), u(\tau)\rangle d\tau ds \\
&\leq \|u\|^2_{L^2(0,T;L^2(U))} + 2\int_0^T \int_s^t \|u'(\tau)\|_{H^{-1}(U)} \|u(\tau)\|_{H^1(U)} d\tau \\
&\leq \|u\|^2_{L^2(0,T;L^2(U))} + 2T\int_0^T \|u'(\tau)\|_{H^{-1}(U)} \|u(\tau)\|_{H^1(U)} d\tau \\
&\leq \|u\|^2_{L^2(0,T;L^2(U))} + T\int_0^T \|u'(\tau)\|^2_{H^{-1}(U)} + \|u(\tau)\|^2_{H^1(U)} d\tau \\
&= \|u\|^2_{L^2(0,T;L^2(U))} + T\left(\|u'\|^2_{L^2(0,T;H^{-1}(U))} + \|u\|^2_{L^2(0,T;H^1_0(U))}\right)
\end{aligned}$$

Proof of Theorem 5.9.3

Since the right-hand side does not depend on t, we can compute the maximum over $t \in [0, T]$ on both sides to see that (also dividing by T and taking a square root)

$$\begin{aligned} \|u\|_{C([0,T];L^2(U))} &\leq \tilde{C}\Big(\|u\|^2_{L^2(0,T;H^1_0(U))} + \|u'\|^2_{L^2(0,T;H^{-1}(U))}\Big)^{1/2} \\ &\leq \tilde{C}\Big(\|u\|_{L^2(0,T;H^1_0(U))} + \|u'\|_{L^2(0,T;H^{-1}(U))}\Big), \end{aligned}$$

and this is the claim. □