Second Order Parabolic PDE: Weak Solutions and Galerkin Approximations

MATH 612, Texas A&M University

Spring 2020

Second Order Parabolic PDE

Let $U \subset \mathbb{R}^n$ be open and bounded, and for $T > 0$ set

$$U_T := U \times (0, T].$$

We'll consider equations of the form

$$\begin{aligned} u_t + Lu &= f; \quad \text{in } U_T \\ u &= 0; \quad \text{on } \partial U \times [0, T] \\ u &= g; \quad \text{on } U \times \{t = 0\}, \end{aligned} \tag{$\mathcal{P}$}$$

where L denotes a partial differential operator either in divergence form

$$Lu := -\sum_{i,j=1}^{n} (a^{ij}(\vec{x}, t) u_{x_i})_{x_j} + \sum_{i=1}^{n} b^i(\vec{x}, t) u_{x_i} + c(\vec{x}, t) u,$$

or non-divergence form

$$Lu := -\sum_{i,j=1}^{n} a^{ij}(\vec{x}, t) u_{x_i x_j} + \sum_{i=1}^{n} b^i(\vec{x}, t) u_{x_i} + c(\vec{x}, t) u.$$

Uniform Parabolicity

Definition. For either form of the operator L, we say that the partial differential operator $\partial_t + L$ is uniformly parabolic in U_T if there exists a constant $\theta > 0$ so that

$$\sum_{i,j=1}^{n} a^{ij}(\vec{x}, t)\xi_i\xi_j \geq \theta|\vec{\xi}|^2$$

for a.e. $(\vec{x}, t) \in U_T$ and all $\vec{\xi} \in \mathbb{R}^n$. I.e., if we denote $A(\vec{x}, t) = (a^{ij}(\vec{x}, t))$, then

$$\vec{\xi}^T A(\vec{x}, t)\vec{\xi} \geq \theta|\vec{\xi}|^2,$$

for a.e. $(\vec{x}, t) \in U_T$ and all $\vec{\xi} \in \mathbb{R}^n$.

Notes. 1. For the heat equation, $L = -\Delta$, so that $A(\vec{x}, t) = I$, and we have

$$\vec{\xi}^T A(\vec{x}, t)\vec{\xi} = |\vec{\xi}|^2.$$

I.e., $\theta = 1$.

Uniform Parabolicity

2. More generally, if $A(\vec{x}, t)$ is symmetric, then by the min-max principle, this condition holds if and only if the eigenvalues of $A(\vec{x}, t)$ are all bounded below by θ. Recall from last semester that a partial differential operator $\partial_t + L$ is parabolic if the matrix associated with its second order terms (including t) has 0 as one of its eigenvalues and its other eigenvalues all have the same sign. So an operator that is uniformly parabolic in U_T is certainly parabolic in U_T.

Basic Assumptions

In order to avoid repeated statements of assumptions, we'll collect our basic assumptions for this section here. These will be:

1. $U \subset \mathbb{R}^n$ is open and bounded, and $T > 0$;
2. $a^{ij} = a^{ji}$ for all $i, j \in \{1, 2, \dots, , n\}$;
3. a^{ij}, b^i, $c \in L^\infty(U_T)$ for all $i, j \in \{1, 2, \dots, , n\}$;
4. The operator L is in divergence form and uniformly parabolic in U_T;
5. $f \in L^2(U_T)$, $g \in L^2(U)$.

We'll refer to this collection of assumptions as Assumptions **(A)**.

Note. In this section, unless explicitly stated otherwise, $(\cdot, \cdot)$ will denote $L^2(U)$ inner product, and $\langle \cdot, \cdot \rangle$ will denote the action of an element of $H^{-1}(U)$ on an element of $H_0^1(U)$.

The Time-Dependent Bilinear Form

Formally, if u is a smooth solution of

$$u_t + Lu = f,$$

with L in divergence form, we can multiply by a test function $\phi \in C_c^\infty(U)$ and proceed as in our section on elliptic operators to obtain the relation

$$(u_t, \phi) + B[u, \phi; t] = (f, \phi),$$

for each $t \in [0, T]$, where

$$B[u, \phi; t] := \int_U \Big\{ \sum_{i,j=1}^n a^{ij}(\vec{x}, t) u_{x_i} \phi_{x_j} + \sum_{i=1}^n b^i(\vec{x}, t) u_{x_i} \phi + c(\vec{x}, t) u \phi \Big\} d\vec{x}.$$

We'll use this to develop our weak formulation for $(\mathcal{P})$.

Notation

If $u(\cdot, t) \in H_0^1(U)$ for a.e. $t \in (0, T)$, then we'll regard u as a map $t \mapsto u(\cdot, t)$. In this section, following the convention that Evans adopts, we'll denote such maps with a bold $\mathbf{u}$. I.e.,

$$\mathbf{u}(t)(\vec{x}) = u(\vec{x}, t),$$

and similarly for $\mathbf{f}(t)$. This allows us to express the strong form of $(\mathcal{P})$ as

$$\mathbf{u}' + L\mathbf{u} = \mathbf{f}, \quad \text{for } a.e.\, t \in (0, T).$$

Here, we notice that since $f \in L^2(U_T)$, we have

$$\|\mathbf{f}\|^2_{L^2(0,T;L^2(U))} = \int_0^T \|\mathbf{f}(t)\|^2_{L^2(U)} dt = \int_0^T \int_U |f(\vec{x}, t)|^2 d\vec{x} dt < \infty.$$

I.e., $\mathbf{f} \in L^2(0, T; L^2(U))$.

Motivating the Weak Formulation

The associated weak form of this equation can be expressed as

$$(\mathbf{u}', v) + B[\mathbf{u}, v; t] = (\mathbf{f}, v),$$

for all $v \in H_0^1(U)$ and a.e. $t \in (0, T)$. We see that

$$\begin{aligned}
(\mathbf{u}', v) &= -B[\mathbf{u}, v; t] + (\mathbf{f}, v) \\
&= -\int_U \left\{ \sum_{i,j=1}^{n} a^{ij} \mathbf{u}_{x_i} v_{x_j} + \sum_{i=1}^{n} b^i \mathbf{u}_{x_i} v + c\mathbf{u} v \right\} + (\mathbf{f}, v) \\
&= \int_U \left\{ \left(\mathbf{f} - \sum_{i=1}^{n} b^i \mathbf{u}_{x_i} - c\mathbf{u} \right) v - \sum_{j=1}^{n} \left(\sum_{i=1}^{n} a^{ij} \mathbf{u}_{x_i} \right) v_{x_j} \right\} d\vec{x} \\
&= \int_U \left\{ \mathbf{g}^0 v + \sum_{j=1}^{n} \mathbf{g}^j v_{x_j} \right\} d\vec{x}.
\end{aligned}$$

Motivating the Weak Formulation

I.e.,

$$(\mathbf{u}', v) = \int_U \left\{ \mathbf{g}^0 v + \sum_{j=1}^n \mathbf{g}^j v_{x_j} \right\} d\vec{x},$$

where for $\mathbf{u}(t) \in H_0^1(U)$, we have

$$\mathbf{g}^0(t) = \mathbf{f}(t) - \sum_{i=1}^n b^i(\vec{x}, t)\mathbf{u}(t)_{x_i} - c(\vec{x}, t)\mathbf{u}(t) \in L^2(U)$$

$$\mathbf{g}^j(t) = -\sum_{i=1}^n a^{ij}(\vec{x}, t)\mathbf{u}_{x_i} \in L^2(U).$$

Motivating the Weak Formulation

If we compare with Theorem 5.9.1, we see that this suggests that we should have $\mathbf{u}'(t) \in H^{-1}(U)$ for a.e. $t \in (0, T)$, with

$$\begin{aligned} \|\mathbf{u}'(t)\|_{H^{-1}(U)} &\leq \Big(\int_U \sum_{j=0}^{n} |\mathbf{g}^j(t)|^2 d\vec{x} \Big)^{1/2} \\ &\leq C\Big(\|\mathbf{u}(t)\|_{H^1(U)} + \|\mathbf{f}(t)\|_{L^2(U)} \Big). \end{aligned}$$

This motivates our weak formulation of $(\mathcal{P})$.

The Parabolic Weak Formulation

Definition. We say that a function $\mathbf{u} \in L^2(0, T; H_0^1(U))$, with $\mathbf{u}' \in L^2(0, T; H^{-1}(U))$, is a weak solution of $(\mathcal{P})$ if

(i) $\langle \mathbf{u}', v \rangle + B[\mathbf{u}, v; t] = (\mathbf{f}, v)$ for all $v \in H_0^1(U)$ and a.e. $t \in (0, T)$, and

(ii) $\mathbf{u}(0) = g$.

Note. We know from Theorem 5.9.3 that under these assumptions, we have $\mathbf{u} \in C([0, T]; L^2(U))$, so the pointwise evaluation $\mathbf{u}(0) = g$ is justified.

Galerkin Approximations

We'll approach existence by first constructing solutions to certain finite-dimensional approximations of the weak formulation of $(\mathcal{P})$, and then taking an appropriate limit. This method is named after the Russian mathematician Boris Galerkin (1871-1945).

To begin, let $\{w_k\}_{k=1}^{\infty}$ denote an orthogonal basis of $H_0^1(U)$ that is also an orthonormal basis of $L^2(U)$. For example, such a basis is constructed in the proof of Theorem 6.5.2 in Evans as eigenfunctions for the Laplacian operator $L = -\Delta$ on U.

We fix any $m \in \{1, 2, \dots\}$, and look for solutions of the weak formulation of $(\mathcal{P})$ of the form

$$\mathbf{u}_m(t) = \sum_{k=1}^{m} d_m^k(t) w_k,$$

where the coefficient functions $\{d_m^k(t)\}_{k=1}^{m}$ are to be determined.

Galerkin Approximations

If we substitute this ansatz into the weak formulation of $(\mathcal{P})$, we obtain the relation

$$\langle \sum_{k=1}^{m} d_m^{k\,\prime}(t) w_k, v \rangle + B[\sum_{k=1}^{m} d_m^k(t) w_k, v; t] = (\mathbf{f}(t), v), \quad \forall\, v \in H_0^1(U),$$

and using linearity

$$\sum_{k=1}^{m} d_m^{k\,\prime}(t) \langle w_k, v \rangle + \sum_{k=1}^{m} d_m^k(t) B[w_k, v; t] = (\mathbf{f}(t), v), \quad \forall\, v \in H_0^1(U).$$

Here, we have $w_k \in H_0^1(U)$ for each fixed $k \in \{1, 2, \ldots, m\}$, so we can replace $\langle w_k, v \rangle$ with $(w_k, v)_{L^2(U)}$.

Galerkin Approximations

In fact, since $\mathbf{u}_m(t)$ is a finite-dimensional approximation of the solution, we expect that this is too much to ask, but we can think of replacing the requirement that this be true for all $v \in H_0^1(U)$ with the requirement that it be true for all $v \in \mathrm{Span}\{w_j\}_{j=1}^m$. I.e., we require

$$\sum_{k=1}^m d_m^{k\,\prime}(t)(w_k, w_j) + \sum_{k=1}^m d_m^k(t) B[w_k, w_j; t] = (\mathbf{f}(t), w_j),$$

for all $j \in \{1, 2, \dots, m\}$. We set

$$e^{jk}(t) := B[w_k, w_j; t] \quad \text{and} \quad f^j(t) := (\mathbf{f}(t), w_j),$$

and use orthonormality of $\{w_k\}_{k=1}^n$ to obtain the first-order system of ODE

$$d_m^{j\,\prime}(t) = -\sum_{k=1}^m e^{jk}(t) d_m^k(t) + f^j(t), \quad j = 1, 2, \dots m.$$

Galerkin Approximations

For initial values, we would like to set

$$\mathbf{u}_m(0) = g \in L^2(U),$$

but again this is too much to ask. Instead, we formally determine the coefficients $\{d_m^k(0)\}_{k=1}^m$ that we would need in order to have the relation

$$g = \sum_{k=1}^{\infty} d_m^k(0) w_k.$$

Again using orthonormality of $\{w_k\}_{k=1}^{\infty}$, we see that

$$d_m^j(0) = (g, w_j),$$

for each $j \in \{1, 2, \dots, m\}$.

Galerkin Approximations

We now want to assert something about solvability for the ODE system

$$d_m^{j\,\prime}(t) = -\sum_{k=1}^{m} e^{jk}(t) d_m^k(t) + f^j(t), \quad j = 1, 2, \ldots m$$
$$d_m^j(0) = (g, w_j),$$

and for this we need to better understand the nature of the coefficents $\{f^j(t)\}_{j=1}^m$ and $\{e^{jk}(t)\}_{j=1}^m$. First,

$$\|f^j\|^2_{L^2(0,T)} = \int_0^T |(\mathbf{f}(t), w_j)|^2 dt \overset{c.s.}{\leq} \int_0^T \|\mathbf{f}(t)\|^2_{L^2(U)} \|w_j\|^2_{L^2(U)} dt$$
$$= \|\mathbf{f}\|_{L^2(0,T;L^2(U))} < \infty,$$

so $f^j \in L^2(0, T; \mathbb{R})$ for all $j \in \{1, 2, \ldots, m\}$.

Galerkin Approximations

Likewise, for each e^{jk},

$$\|e^{jk}\|_{L^\infty(0,T)} = \|B[w_k, w_j; t]\|_{L^\infty(0,T)},$$

and

$$\Big|B[w_k, w_j; t]\Big| = \Big| \int_U \Big\{ \sum_{i,l=1}^{n} a^{il}(w_k)_{x_j}(w_j)_{x_l} + \sum_{i=1}^{n} b^i(w_k)_{x_i} w_j + c w_k w_j \Big\} d\vec{x} \Big|$$

$$\leq C \int_U \Big\{ \sum_{i,l=1}^{n} |(w_k)_{x_j}||(w_j)_{x_l}| + \sum_{i=1}^{n} |(w_k)_{x_i}||w_j| + |w_k||w_j| \Big\} d\vec{x}$$

$$\leq \tilde{C}\|w_k\|_{H^1(U)}\|w_j\|_{H^1(U)} < \infty,$$

so $e^{jk} \in L^\infty(0, T; \mathbb{R})$ for all $j, k \in \{1, 2, \ldots, m\}$.

Galerkin Approximations

In summary, for the ODE system

$$d_m^{j\prime}(t) = -\sum_{k=1}^{m} e^{jk}(t) d_m^k(t) + f^j(t), \quad j = 1, 2, \dots m$$
$$d_m^j(0) = (g, w_j),$$

we have $f^j \in L^2(0, T; \mathbb{R})$ and $e^{jk} \in L^\infty(0, T; \mathbb{R})$ for all $j, k \in \{1, 2, \dots, m\}$, which is more than we need for what we want to assert. Proceeding similarly as we did first semester, we can show that as long as $f^j, e^{jk} \in L^1(0, T; \mathbb{R})$, then there exists a unique absolutely continuous solution

$$\mathbf{d}_m(t) = (d_m^1(t), d_m^2(t), \dots, d_m^m(t))$$

to this system on $[0, T]$.

For details on this existence, see, e.g., Theorem 2.1 in "Spectral Theory of Ordinary Differential Operators," by Joachim Weidmann, Lecture Notes in Mathematics **1258** (1987).

Galerkin Approximations

By construction, we see that

$$\mathbf{u}_m(t) = \sum_{k=1}^{m} d_m^k(t) w_k, \qquad \text{(G)}$$

solves the finite-dimensional weak problem,

$$\langle \mathbf{u}_m', w_j \rangle + B[\mathbf{u}_m, w_j; t] = (\mathbf{f}, w_j), \quad \forall\, j \in \{1, 2, \dots, m\}, \qquad \text{(FDW)}$$

for a.e. $t \in (0, T)$, along with

$$d_m^k(0) = (g, w_k), \quad \forall\, k \in \{1, 2, \dots, m\}. \qquad \text{(IC)}$$

In this way, we have established the following theorem from Evans:

Theorem 7.1.1. Let Assumptions **(A)** hold. Then for each integer $m \in \{1, 2, \dots\}$, there exists a unique function $\mathbf{u}_m$ of the form (G) solving (FDW), (IC).

Galerkin Approximations

We refer to $\mathbf{u}_m(t)$ constructed in this way as the Galerkin approximation of our sought solution $\mathbf{u}(t)$. Our goal will be to show that as $m \to \infty$, the sequence $\{\mathbf{u}_m\}_{m=1}^{\infty}$ converges to a solution of the weak formulation of $(\mathcal{P})$ in an appropriate sense.