Note Title 8/2/2015

Application to Linear Systems

In this application, we're considering equations of the form

$$Ax = b,$$

where A is an $n \times n$ matrix and $x, b \in \mathbb{R}^n$.

In a previous lecture we wrote $A = A_1 + A_2$ and derived the iteration

$$x_{n+1} = A_1^{-1} b - A_1^{-1} A_2 x_n .$$

This is a recursion $x_{n+1} = T(x_n)$ with

$$T(x) = A_1^{-1} b - A_1^{-1} A_2 x .$$

We can check if T is a contraction near a fixed point x^* by looking at the eigenvalues of

$$(DT(x^*))^{tr} \, DT(x^*) .$$

In this case

$$DT(x) = -A_1^{-1} A_2$$

and so of course

$$DT(x^*) = -A_1^{-1} A_2$$

We have, then

$$(DT(x^*))^{tr} DT(x^*) = (-A_1^{-1}A_2)^{tr} (-A_1^{-1}A_2)$$

$$= A_2^{tr} (A_1^{-1})^{tr} A_1^{-1} A_2 .$$

If the eigenvalues of this matrix are all less than 1, then T is a contraction. Recall that there are different ways to choose A_1 and A_2, and one goal is to choose them to make these eigenvalues as small as possible (certainly less than 1).

Students may have notice that

$$|T(x) - T(y)| = |A_1^{-1}b - A_1^{-1}A_2 x - (A_1^{-1}b - A_1^{-1}A_2 y)|$$
$$= |A_1^{-1}A_2(y-x)|$$

$$\le |A_1^{-1}A_2||x-y|,$$

as long as we know what we mean by $|A_1^{-1}A_2|$, which is the norm of a matrix. One way to measure this norm is to take the square root of the largest eigenvalue of this matrix we've been constructing: $(A_1^{-1}A_2)^{tr}A_1^{-1}A_2$.