Note Title 7/28/2015

Series Characterization of Banach Spaces

Theorem 7.12

A normed vector space X is complete (i.e. is a Banach space) iff every absolutely summable series in X is summable.

Proof

First, for $(\Rightarrow)$ we suppose X is complete and that we have a series that is absolutely summable. We need to check that the series is summable. To see this, consider the partial sums

$$S_m = \sum_{k=1}^{m} x_k,$$

and notice that for $m > n$

$$\| S_m - S_n \| = \| \sum_{k=n+1}^{m} x_k \| \leq \sum_{k=n+1}^{m} \| x_k \| \leq \sum_{k=n+1}^{\infty} \| x_k \|$$

We know the series $\sum_{k=1}^{\infty} \|x_k\|$ converges, and so $\sum_{k=n+1}^{\infty} \|x_k\| \to 0$ as $n \to \infty$. But this means the sequence of partial sums (S_n) is Cauchy, and so it converges (because of completeness). This convergence of partial sums means, by definition, that the series is summable.

For ($\Leftarrow$) we suppose every absolutely summable series is summable, and we let (x_n) be any Cauchy sequence in X. We need to show that (x_n) converges in X. As noted in Problem 1.15, it's enough to show that (x_n) has a convergent subsequence.

Since (x_n) is Cauchy we can take $\|x_n - x_m\|$ as small as we like by taking m and n

sufficiently large. We construct a subsequence as follows: Take N_1 large enough so that

$$n, m \geq N_1 \implies \|x_n - x_m\| < 2^{-1}$$

and fix $y_1 = x_{N_1}$. Next, choose N_2 large enough so that

$$n, m \geq N_2 \implies \|x_n - x_m\| < 2^{-2}$$

Set $y_2 = x_{N_2}$. Notice that $\|y_2 - y_1\| < 2^{-1}$.

In this way, we construct a subsequence $(y_n) \subset (x_n)$ (Carothers denotes (y_n) by (x_{n_k})) so that

$$\|y_{n+1} - y_n\| < 2^{-n}$$

for all $n = 1, 2, \ldots$

We have then

$$\sum_{n=1}^{\infty} \|y_{n+1} - y_n\| < \sum_{n=1}^{\infty} 2^{-n} = 1,$$

and so the sum of the left-hand side converges. (I.e., the partial sums

$$S_N = \sum_{n=1}^{N} \| y_{n+1} - y_n \|$$

are monotonically increasing and bounded above, and so converge.)

By assumption, this means $\sum_{n=1}^{\infty} (y_{n+1} - y_n)$ converges, and recall that

$$\sum_{n=1}^{\infty} (y_{n+1} - y_n) = \lim_{n \to \infty} y_n - y_1$$

which means (y_n) converges to $y \in X$. In fact,

$$\lim_{n \to \infty} y_n = y_1 + \sum_{n=1}^{\infty} (y_{n+1} - y_n). \quad \square$$