Note Title 7/29/2015

Matrix Equations

Consider the equation

$$Ax = b,$$

where A is an $n \times n$ matrix, and $x, b \in \mathbb{R}^n$.
Generally, it's difficult (and computationally inefficient) to invert A directly, and one approach is to split A up into a part that is easy to invert and a remainder:

$$A = \underset{\substack{\uparrow \\ \text{easy to} \\ \text{invert}}}{A_1} + \underset{\substack{\uparrow \\ \text{remainder}}}{A_2}$$

As a starting point, observe that every square matrix A can be expressed as

$$A = D + L + U,$$

where D is a diagonal matrix, L is the matrix of values below the diagonal of A, and U is the matrix of values above the diagonal of A. One obvious choice is $A_1 = D$, $A_2 = L + U$ (this leads to Jacobi iterates), and another is $A_1 = D + U$, $A_2 = L$ (this leads to Gauss-Seidel iterates).

For any choice of A_1 and A_2 our equation becomes

$$A_1 x + A_2 x = b,$$

which we can rearrange as

$$A_1 x = b - A_2 x.$$

We now let x_1 be an initial approximation of our solution, and we update it with

$$A_1 x_2 = b - A_2 x_1.$$

This leads to the iteration

$$A_1 x_{n+1} = b - A_2 x_n \qquad n = 1, 2, \dots .$$

We can express this as

$$x_{n+1} = A_1^{-1} b - A_1^{-1} A_2 x_n,$$

which has the form

$$x_{n+1} = T(x_n)$$

of our general recursion equation. The solutions we're looking for are fixed points of this equation.