Note Title 8/1/2015

Proof of the Contraction Mapping Theorem

__Theorem 7.13__ (Contraction Mapping Theorem)

Let (M,d) be a complete metric space, and let $T: M \to M$ be a (strict) contraction. Then T has a unique fixed point x. Moreover, given any point $x_0 \in M$

$$\lim_{n\to\infty} T^n(x_0) = x.$$

Proof

Let $x_0 \in M$ and consider the sequence

$$(x_n) = (T^n(x_0))$$

Since (M,d) is complete we can verify that this sequence converges by verifying that it's Cauchy. To see this, first notice the following:

$$
\begin{aligned}
d(x_{n+1}, x_n) &= d(T^{n+1}(x_0), T^n(x_0)) \\
&= d(T^n(x_1), T^n(x_0)) \\
&\leq \alpha\, d(T^{n-1}(x_1), T^{n-1}(x_0)) \\
&\;\vdots \\
&\leq \alpha^n d(x_1, x_0).
\end{aligned}
$$

But then for $m \geq n$,

$$
d(x_m, x_n) \leq d(x_{n+1}, x_n) + d(x_{n+2}, x_{n+1}) + d(x_{n+3}, x_{n+2}) + \cdots + d(x_m, x_{m-1})
$$

$$= \sum_{k=n}^{m} d(x_{k+1}, x_k) \leq \sum_{k=n}^{m} \alpha^k d(x_1, x_0)$$

$$= d(x_1, x_0) \sum_{k=n}^{m} \alpha^k .$$

As with a calculation earlier this semester with geometric series, it's easy to check

$$\sum_{k=n}^{\infty} \alpha^k = \frac{\alpha^n}{1-\alpha} .$$

We see that

$$d(x_m, x_n) \leq \frac{\alpha^n}{1-\alpha} d(x_1, x_0)$$

Since $0 \leq \alpha < 1$ and $d(x_1, x_0)$ is fixed, we can make this as small as we like by taking n large, so (x_n) is Cauchy.

We can conclude that (x_n) converges to an element of M, which we'll denote x. I.e., $x_n \to x$ in M. We need to check that x

is a fixed point of T. But this is easy to see, since T is continuous (notice it's clear that contractions are continuous).

That is, we can compute

$$T(x) = T\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} T(x_n)$$

$$= \lim_{n\to\infty} T^{n+1}(x_0) = \lim_{n\to\infty} x_{n+1} = x.$$

Finally, to check uniqueness, suppose x and y are two different fixed points of T so that

$$T(x) = x$$

$$T(y) = y.$$

Then $d(x,y) = d(T(x), T(y)) \le \alpha \, d(x,y)$

$\Rightarrow d(x,y)(1-\alpha) \le 0 \Rightarrow d(x,y) \le 0$

$\Rightarrow d(x,y) = 0 \Rightarrow x = y.$ $\square$

We can use one step of this proof to get an estimate for how close approximation x_n is to actual fixed point x. We have

$$d(x_m, x_n) \leq \frac{\alpha^n}{1-\alpha} d(x_1, x_0)$$

for $m \geq n$. Taking $m \to \infty$ we see that

$$\boxed{d(x, x_n) \leq \frac{\alpha^n}{1-\alpha} d(x_1, x_0)}$$