Contraction Maps on $\mathbb{R}$ and $\mathbb{R}^n$

Note Title 8/1/2015

Suppose we would like to show that $T: \mathbb{R} \to \mathbb{R}$ is a contraction map. We'll assume that T has a fixed point, which we'll denote x^*, and we'll assume that T is continuously differentiable in a neighborhood of x^*. According to the Mean Value Theorem we can always write

$$T(x) - T(y) = T'(z)(x-y)$$

for some z between x and y. This clearly gives the inequality

$$|T(x) - T(y)| \leq |T'(z)|\,|x-y|.$$

Suppose we can show that $|T'(x^*)| < 1$. By continuity we'll know $|T'(z)| < 1$ for z sufficiently close to x^*. In particular, suppose that for some $0 \leq \alpha < 1$

$$|T'(z)| \leq \alpha$$

on some interval

$$I = (x^* - \delta,\ x^* + \delta),\quad \delta > 0.$$

(So $|T(x) - T(y)| \leq \alpha |x - y| \quad \forall x, y \in I.$)

Now suppose our initial approximation x_0 is good enough so that $x_0 \in I$ (i.e., $|x_0 - x^*| < \delta$). Our next approximation will be $x_1 = T(x_0)$, and so x^* is a fixed point

$$|x_1 - x^*| = |T(x_0) - T(x^*)|$$
$$\leq \alpha |x_0 - x^*| < \alpha \delta .$$

We see that $x_1 \in I$. Likewise, $x_2 \in I$, etc., so that $(x_n) \subset I$. We can conclude that T is a contraction map for the full iteration.

This shows: If $|T'(x^*)| < 1$ then if x_0 is chosen close enough to x^*, the iteration $x_{n+1} = T(x_n)$ will converge to x^*.

The analysis is more complicated for $T: \mathbb{R}^n \to \mathbb{R}^n$, but we can summarize the result as follows: if $DT(x^*)$ denotes the Jacobian matrix of T at the fixed point x^*, and the eigenvalues of

$$\left(DT(x^*)\right)^{tr} DT(x^*)$$

transpose

(which are all non-negative) are all less than 1, then T is a contraction map near x^*, so if x_0 is chosen close enough to x^* then the recursion $x_{n+1} = T(x_n)$ will converge to x^*.