Application to Newton's Method

Note Title 8/1/2015

Recall that our goal is to find a root x^* of an equation

$$f(x) = 0.$$

We start with an initial approximation x_0, and iterate with

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

This is a recursion relation

$$x_{n+1} = T(x_n),$$

where

$$T(x) = x - \frac{f(x)}{f'(x)}.$$

The method will converge if T is a contraction, and we've seen that T will be a contraction if $|T'(x^*)| < 1$. (This assumes x_0 is a sufficiently close first approximation.)

We compute

$$T'(x) = 1 - \frac{f'(x)^2 - f(x)f''(x)}{f'(x)^2}$$

$$= \underbrace{1 - 1}_{0} + \frac{f(x)f''(x)}{f'(x)^2}.$$

Since $f(x^*) = 0$ ($\because x^*$ is a root) we see that $T'(x^*) = 0$. We have $|T'(x^*)| < 1$, so we can conclude that

Newton's method for a single equation always converges for a sufficiently good initial approximation.

For a system with $x, f \in \mathbb{R}^n$, Newton's method takes the form

$$x_{n+1} = x_n - (Df(x_n))^{-1} f(x_n).$$

In this case,

$$T(x) = x - (Df(x))^{-1} f(x).$$

We compute

$$DT(x) = \underline{I} - D(Df(x))^{-1} f(x) - \underbrace{(Df(x))^{-1} Df(x)}_{\underline{I}}$$

$$= -D(Df(x))^{-1} f(x)$$

Since $f(x^*) = 0$, we see that $DT(x^*) = 0$. This means

$$(DT(x^*))^{tr} DT(x^*)$$

is the 0 matrix, and so its eigenvalues are all 0. But we know that if the eigenvalues of $DT(x^*)$ for $DT(x^*)$ are all less than 1, T is a contraction (near x^*). So we see that for systems, as with single equations, Newton's method always converges with a sufficiently good initial guess.