Application to ODEs, II

Note Title 8/2/2015

We saw in Part I that we need to show that the map

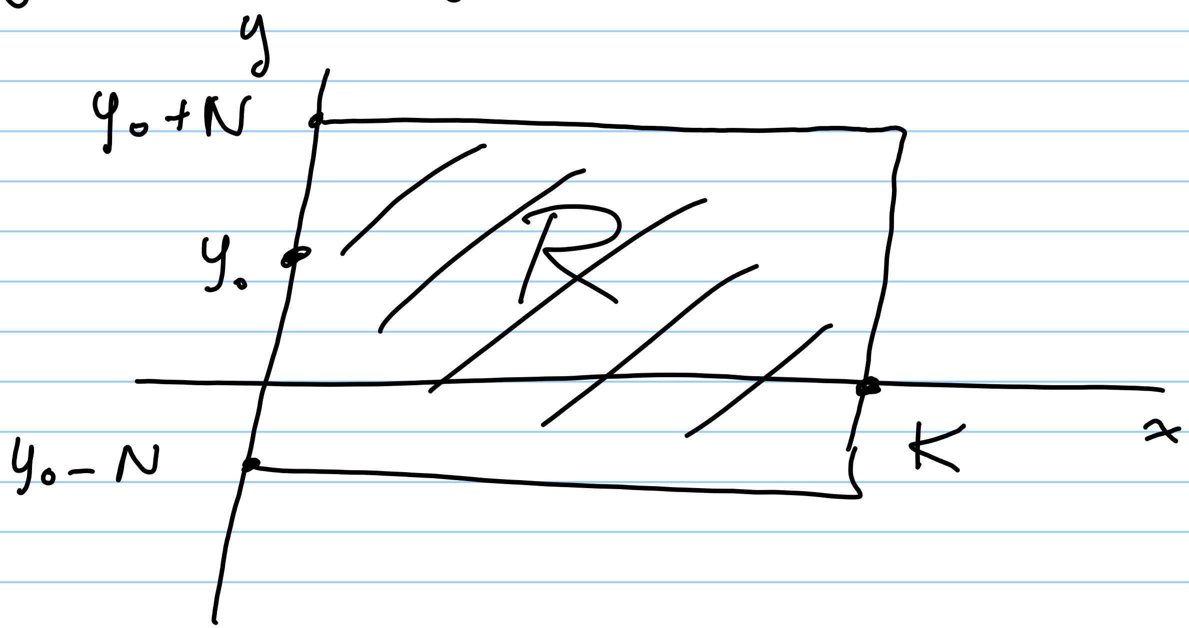
$$T(y) = y_0 + \int_0^x f(t, y(t))\,dt$$

is a contraction. This will require some assumptions on f. We'll take $y, f \in \mathbb{R}^n$, and use $|\cdot|$ for Euclidean norm.

We'll assume f is continuous in a region

$$R := \{(x,y) : 0 \le x \le K,\ |y - y_0| \le N\}.$$

Figure for $y \in R$.



Set $\quad C = \max_{(x,y)\in R} |f(x,y)|$.

We also assume f is Lipschitz continuous in y, uniformly in x:

$$|f(x,y_1) - f(x,y_2)| \le L\,|y_1 - y_2|$$

for all $(x,y_1), (x,y_2) \in R$.

Next, we need to define an appropriate metric space.

For $a := \min\left\{K, \frac{N}{C}, \frac{1}{2L}\right\}$, we set

$$M = \left\{ y \in C([0,a]) : y(0) = y_0,\ \|y - y_0\| \leq N \right\},$$

where $\|\cdot\|$ denotes the norm

$$\|y\| = \max_{x \in [0,a]} |y(x)|$$

As a start, we need to show $T : M \to M$.
(This is called invariance.) I.e., given $g \in M$,
we need to show $T(g) \in M$.

Recall $T(y) = y_0 + \int_0^x f(t, y(t))\,dt$.

First, $T(y)$ is differentiable, and so it is certainly continuous. Second $T(y)(0) = y_0$.

Third,

$$T(y) - y_0 = \int_0^x f(t, y(t))\,dt$$

$$\Rightarrow |T(y) - y_0| \leq \int_0^x |f(t, y(t))|\,dt$$

$$\leq aC$$

Since the right-hand side is independent of x, we can take $\max_{x\in[0,1]}$ on both sides to get

$$\|T(y)-y_0\| \le aC.$$

But we've taken $a \le \frac{N}{C} \Rightarrow$

$$\|T(y)-y_0\| \le N.$$

Now, to see that T is a contraction, we compute

$$T(y_1) - T(y_1) = \int_0^x f(t, y_1(t)) - f(t, y_2(t))\, dt$$

$$\Rightarrow \left| T(y_1) - T(y_2) \right| \leq \int_0^x \left| f(t, y_1(t)) - f(t, y_2(t)) \right| dt$$

$$\leq \int_0^x L \left| y_1(t) - y_2(t) \right| dt$$

$$\leq a L \| y_1 - y_2 \| .$$

Take $\max\limits_{x \in [0,a]}$ to see

$$\|T(y_1) - T(y_2)\| \le aL\,\|y_1 - y_2\|$$

We need $aL < 1$, and recall we've assumed $a \le \frac{1}{2L} \Rightarrow aL \le \frac{1}{2} < 1$.

We conclude that T is a contraction on M, and so there exists a unique solution $y = T(y)$:
$$y(x) = y_0 + \int_0^x f(t, y(t))\,dt.$$

Differentiating, we find

$$\frac{dy}{dx} = f(x, y(x)); \qquad y(0) = y_0,$$

and this is the equation we started with.