Note Title 8/4/2015

The Space $\ell_\infty(M)$

Here's the goal: Given a metric space (M,d) that is not complete, we need to identify a metric space $(\hat{M},\hat{d})$ that is complete, and that isometrically contains (M,d). One way to proceed is by considering the collection of all bounded real-valued functions defined on M. That is, the space of functions

$$f : M \to \mathbb{R}$$

so that the norm

$$\|f\|_\infty := \sup_{x \in M} |f(x)|$$

is bounded. We denote this space $l_\infty(M)$.

Notice that this is consistent with our previous use of l_∞, since a bounded sequence of real numbers can be viewed as a bounded

function on $\mathbb{N}$. In other words $l_\infty = l_\infty(\mathbb{N})$.
Let's denote the associated metric d_∞, so

$$d_\infty(f,g) = \| f - g \|_\infty .$$

A good example to keep in mind is $M = (0,1)$, which is not complete, because there are Cauchy sequences that converge to 0 and 1, which are not in the set. In this case, $l_\infty(0,1)$ is the collection of bounded real-valued functions defined on $(0,1)$.

It's clear that $\|\cdot\|_\infty$ is a norm (i.e., the norm properties are satisfied). It turns out that $l_\infty(M)$ is complete. The proof is similar to the proof that l_∞ (i.e. $l_\infty(\mathbb{N})$) is complete, and is assigned in Problem 7.44.

We have: $(\ell_\infty(M), d_\infty)$ is a complete metric space.