M is Isometric to a Subset of $l_\infty(M)$

Note Title 8/4/2015

Given a metric space (M,d), which we assume is not complete (otherwise it could be its own completion), we have an associated metric space $(l_\infty(M), d_\infty)$, which is complete.

In our lemma, we'll show that (M,d) is isometric to a subset of $(l_\infty(M), d_\infty)$ (though not necessarily a dense subset).

For this, we need to find a 1–1 map

$$i : M \to l_\infty(M)$$

so that for all $x, y \in M$

$$d(x,y) = d_\infty(i(x), i(y)).$$

Then $i : M \to i(M)$ will be an isometry onto the subset $i(M) \subset l_\infty(M)$.

Lemma 7.17

Let (M, d) be any metric space. Then M is isometric to a subset of $l_\infty(M)$.

Proof

We need to associate with each $x \in M$ a function $f_x(t) \in l_\infty(M)$ so that for all $x, y \in M$

$$d(x,y) = d_\infty(f_x, f_y) = \|f_x - f_y\|_\infty .$$

To start, we fix some $a \in M$ and for each $x \in M$ set

$$f_x(t) := d(x,t) - d(a,t).$$

From a relation related to the triangle inequality (see Problem 3.2)

$$|f_x(t)| = |d(x,t) - d(a,t)| \leq d(x,a).$$

Since we have no t dependence on the right-hand side, we can take $\sup_{t \in M}$ to see

$$\|f_x\|_\infty \le d(x,a).$$

We see that $f_x \in \ell_\infty(M)$, and this is the first half of what we needed to show.

Next, we need to show that this map

$$x \longrightarrow f_x(t)$$

gives an isometry. For this, we have

$$\|f_x - f_y\|_\infty = \sup_{t\in M} \left|\left(d(x,t) - \cancel{d(p,t)}\right) - \left(d(y,t) - \cancel{d(p,t)}\right)\right|$$

$$= \sup_{t\in M} |d(x,t) - d(y,t)|$$

$$\leq \sup_{t\in M} d(x,y)$$

$$= d(x,y).$$

Notice that by setting $t = x$ we see

$$\left. d(x,t) - d(y,t) \right|_{t=x} = -d(y,x)$$

so that

$$\left. \left| d(x,t) - d(y,t) \right| \right|_{t=x} = d(x,y)$$

$\Longrightarrow$

$$\sup_{t \in M} \left| d(x,t) - d(y,t) \right| = d(x,y)$$

This means

$$\| f_x - f_y \|_\infty = d(x,y). \qquad \square$$

We've shown that (M,d) is isometric to

$$\{ f_x : x \in M \} \subset \ell_\infty(M).$$