Note Title 8/5/2015

The Completion of (M,d)

Let's recall our set-up: we fix $a \in M$, and for each $x \in M$, we associate the function $f_x \in \ell_\infty(M)$ defined by

$$f_x(t) = d(x,t) - d(a,t).$$

We've shown that M is isometric to a subset of $\ell_\infty(M)$, $\{f_x : x \in M\}$, but what we need is for M to be isometric to a dense subset of some appropriate complet metric space $(\hat{M}, \hat{d})$.

We propose:

$$\hat{M} = \text{closure of } \{f_x : x \in M\},$$

with $\hat{d} = d_\infty$.

Since $l_\infty(M)$ is complete, $\hat{M}$ is complete (by Theorem 7.9), and by the definition of density $\{f_x : x \in M\}$ is dense in $\hat{M}$. So (M, d) is isometric to a dense subset of $(\hat{M}, \hat{d})$.

$$\{f_x : x \in M\}$$

$\Rightarrow$ by definition $(\hat{M}, \hat{d})$ is the completion of (M,d).

Now that we've completed (M,d), we might ask: could (M,d) have a second completion? I.e., are completions unique? The answer is yes. We'll check in the next lecture that any two completions must be isometric, and so can be viewed as the same space.