Uniqueness of Completions

Note Title 8/5/2015

Theorem 7.18

If (M_1, d_1) and (M_2, d_2) are completions of (M, d), then (M_1, d_1) and (M_2, d_2) are isometric.

Proof

In principle, M is isometric to dense subsets of M_1 and M_2 (by definition of completion).

but for notational simplicity we'll take M to actually <u>be</u> a dense subset of both M_1 and M_2. Nothing is lost in the logic of the proof. Notice that by isometry this means the metrics d_1, d_2, and d will coincide on M.

We need to show that M_1 and M_2 are isometric; i.e., that there exists a 1-1 and onto map $i: M_1 \to M_2$ so that for $x, y \in M_1$

$$d_1(x,y) = d_2(i(x), i(y)).$$

As a start, notice that given any $x \in M_1$ there exists a sequence $(x_n) \subset M$ so that $x_n \to x$ in M_1 (because M is dense in M_1). Since (x_n) converges in M_1, it must be Cauchy in M_1. By isometry, (x_n) is Cauchy in M_1 and since $M \subset M_2$ this means (x_n) is Cauchy in M_2. So $x_n \to y$ for some $y \in M_2$, by completeness of M_2. Our candidate for an isometry will be

$$i(x) = y$$

where x and y are related as above. We need to check that i is an isometry $i : M_1 \to M_2$.

This means we need to check:

1. i is well-defined; i.e. $i(x)$ cannot take two different values.

2. i is onto.

3. i satisfies the isometry property

$$d_2(i(x), i(y)) = d_1(x,y), \quad \forall x,y \in M_1$$

(1-1 follows from this).

For (1), suppose (x_n) and (z_n) are two different sequences converging to $x \in M_1$.
We need to be sure (x_n) and (z_n) both converge to the same element $y \in M_2$.

First, let $x_n \to y$ in M_2, so $d_2(x_n, y) \to 0$ as $n \to \infty$. We need to see that $d_2(z_n, y) \to 0$ as $n \to \infty$. But

$$d_2(z_n, y) \leq d_2(z_n, x_n) + d_2(x_n, y)$$

$$= d_1(z_n, x_n) + d_2(x_n, y)$$

$$\leq \underbrace{d_1(z_n, x) + d_1(x, x_n)}_{\substack{n \to \infty \\ \longrightarrow 0}} + d_2(x_n, y)$$

$\Rightarrow \quad d_2(z_n, y) \to 0 \quad$ as $\quad n \to \infty$.

Notice that if $x \in M$ then the sequence $(x_n) \subset M$ can be taken as $x_n = x \;\; \forall n$, so it's clear that $y = x$. I.e.,

$$i\big|_{M} = I = \text{identity map}.$$

For (2), we need to check that i is onto. Fix any $y \in M_2$, and recall that since M is dense in M_2 there exists a sequence $(y_n) \subset M$ so that $y_n \longrightarrow y$ in M_2. But precisely as in our calculation verifying that i is well-defined we must have $y_n \to x$ in M_1, for some $x \in M_1$. And this means $i(x) = y$.

Finally, for (3), let $x, y \in M_1$ with sequences (x_n) and (y_n) in M so that $x_n \to x$ in M_1 and $y_n \to y$ in M_1. Then by definition of i $x_n \to i(x)$ in M_2, and $y_n \to i(y)$ in M_2.

We claim:

$$d_1(x,y) = \lim_{n\to\infty} d(x_n, y_n)$$

and

$$d_2(i(x), i(y)) = \lim_{n\to\infty} d(x_n, y_n)$$

This will imply $d_1(x,y) = d_2(i(x), i(y))$, completing the proof.

For the first, since $x_n, y_n \in M \quad \forall n$

$$d_1(x_n, y_n) = d(x_n, y_n) \qquad \forall n$$

so by Problem 3.34

$$\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d_1(x_n, y_n) = d_1(x, y)$$

Likewise, for the second

$$d_2(x_n, y_n) = d(x_n, y_n)$$

and by the same homework problem

$$\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d_2(x_n, y_n) = d_2(i(x), i(y)). \quad \square$$