Completions of Normed Vector Spaces

Note Title 8/6/2015

Theorem

Suppose X is a normed vector space. Then its completion $\hat{X}$ is a Banach space.

Proof

As with our proof of Theorem 7.18, we'll assume X is actually a subset of $\hat{X}$ (not just isometric to one). We know $(\hat{X}, \hat{d})$ is a complete metric space, so we only need to verify that $\hat{X}$ is a normed vector space.

For a normed vector space, we need a notion of scalar multiplication and vector addition. I.e., for $\alpha, \beta \in \mathbb{R}$ and $x, y \in \hat{X}$ we need to make sense of $\alpha x + \beta y$. We can do this in the framework of completions by letting

(x_n) and (y_n) be the corresponding Cauchy sequences for x and y and defining

$$\alpha x + \beta y := \lim_{n\to\infty} (\alpha x_n + \beta y_n).$$

Since $\hat{X}$ is complete we see that

$$\alpha x + \beta y \in \hat{X},$$

and this is the primary requirement for a vector space. (The others are easy to verify.)

We also need to supply $\hat{X}$ with a norm.
We know that it has a metric $\hat{d}$, so a natural candidate

$$\|x\|_{\hat{X}} = \hat{d}(x, 0).$$

Since X is a normed vector space, and since d and $\hat{d}$ coincide on X,

$$\hat{d}(x, 0) = \lim_{n \to \infty} \hat{d}(x_n, 0) = \lim_{n \to \infty} d(x_n, 0) = \lim_{n \to \infty} \|x_n\|_X$$

I.e.,

$$\|x\|_{\hat{X}} = \lim_{n\to\infty} \|x_n\|_X .$$

We need to verify that this is actually a norm. The first three norm properties are easy to see (from p. 40), so let's just verify the triangle inequality. We have, for $x, y \in \hat{X}$:

$$\|x+y\|_{\hat{X}} = \lim_{n\to\infty} \|x_n + y_n\|_X$$

$$\leq \lim_{n\to\infty} \left(\|x_n\|_X + \|y_n\|_X \right)$$

$$= \lim_{n\to\infty} \|x_n\|_X + \lim_{n\to\infty} \|y_n\|_X$$

$$= \|x\|_{\hat{X}} + \|y\|_{\hat{X}} \qquad \square$$