Note Title 8/6/2015

Sequential Characterization of Compactness

Theorem 8.2

A metric space (M, d) is compact iff every sequence in M has a subsequence that converges to a point in M.

Proof

For $(\Longrightarrow)$ suppose (M, d) is compact. Then any sequence in M must be totally bounded, because M is, so by Theorem 7.5 any sequence will have a Cauchy subsequence. But (M,d) is complete, and so this sequence will converge to a point in M.

For $(\Leftarrow)$, by Theorem 7.5 (again) if every sequence in M has a convergent (and so Cauchy) subsequence then M is totally bounded. By Problem 1.15, if (x_n) is Cauchy and has a convergent subsequence, then (x_n) converges, and in this way we see that all Cauchy sequences converge. This means (M,d) is complete. $\square$

Corollary 8.3

Let A be a subset of a metric space M. If A is compact, then A is closed in M. If M is compact and A is closed, then A is compact.

Proof

First, suppose (M, d) is a general metric space, not necessarily compact. Let $(x_n) \subset A$

so that $x_n \to x$ in M. Since (x_n) converges it must be Cauchy, and since A is compact (and so complete) this means $x \in A$.

Next, suppose M is compact, and A is closed. Since $A \subset M$ we know that A is totally bounded (see Problem 7.1), and if $(x_n) \subset A$ is Cauchy then $x_n \to x$ in M since M is compact. But A is closed, so $x_n \to x$ in A, which means A is complete. $\square$