Continuous Functions Map Compact Sets to Compact Sets

Note Title 8/7/2015

Theorem 8.4

Let $f:(M,d)\to(N,\rho)$ be continuous. If K is compact in M, then $f(K)$ is compact in N.

Proof

It follows from Theorem 8.2 that one way to prove this is to show that any sequence

$(y_k) \subset f(K)$ has a convergent subsequence.

To see this, take any $(y_k) \subset f(K)$ and notice that for each k there exists $x_k \in K$ so that

$$f(x_k) = y_k.$$

This defines a sequence $(x_k) \subset K$. Now, K is compact by assumption, so (x_k) has a convergent subsequence (x_{k_j}). I.e.,

$$x_{k_j} \longrightarrow x \qquad j \to \infty$$

in K for some $x \in K$. But f is continuous, so the corresponding subsequence

$$(y_{k_j}) = (f(x_{k_j}))$$

satisfies

$$\lim_{j\to\infty} y_{k_j} = \lim_{j\to\infty} f(x_{k_j}) = f(x).$$

I.e., (y_k) has a subsequence (y_{k_j}) that converges to $f(x) \in f(K)$. $\square$

Notice that this theorem implies the Extreme Value Theorem, which asserts that a real-valued function defined on a compact subset of $\mathbb{R}^n$, if it's continuous, achieves both its minimum and maximum. This is because $f(K) \subset \mathbb{R}$ will be compact (i.e., closed and bounded in this case), and so $\inf f(K)$ and $\sup f(K)$

are both achieved. (See Problem 1.4, and use the fact that $f(K)$ is closed.) In fact, by the same argument (see p. 111 in Carothers) this is true for any compact metric space.

Corollary 8.5

Let (M, d) be compact. If $f: M \to \mathbb{R}$ is continuous then f is bounded. Moreover, f attains its maximum and minimum values.

Corollary 8.6

If $f: [a, b] \to \mathbb{R}$ is continuous, then the range of f is a compact interval $[c, d]$ for some $c, d \in \mathbb{R}$.

The proof is assigned as Problem 8.21.

Corollary 8.7

If M is a compact metric space, then

$$\| f \|_\infty = \max_{t \in M} | f(t) |$$

defines a norm on $C(M)$, the vector space of continuous real-valued functions on M.

Proof

This corollary is placed here because Corollary 8.5 guarantees that $\|f\|_\infty$ is defined. The norm properties are easy to verify. For example, for the triangle inequality, we have:

$$\|f+g\|_\infty = \max_{t \in M} |f(t) + g(t)|$$

$$\leq \max_{t \in M} \left(|f(t)| + |g(t)| \right)$$

$$\leq \max_{t \in M} |f(t)| + \max_{t \in M} |g(t)|$$

$$= \|f\|_\infty + \|g\|_\infty \qquad \square$$