Topological Characterization of Compactness, I

Lemma 8.8

In a metric space M, the following are equivalent:

(a) If $\mathcal{G}$ is any collection of open sets in M with $M \subset \bigcup_{G \in \mathcal{G}} G$ then there are finitely many sets $\{G_k\}_{k=1}^{n}$ in $\mathcal{G}$ so that $M \subset \bigcup_{k=1}^{n} G_k$.

(b) If $\mathcal{F}$ is any collection of closed sets in M so that $\bigcap_{k=1}^{n} F_k \neq \emptyset$ for all choices of finitely many $\{F_k\}_{k=1}^{n}$ in $\mathcal{F}$ then

$$\bigcap_{F \in \mathcal{F}} F \neq \emptyset .$$

The proof is assigned as Problem 8.30.
Carothers gives the hint that this is an application of De Morgan's Laws of set

union and intersection. (See our preliminary lectures).

Notice that Condition (a) implies that M is totally bounded. This is because given any $\varepsilon > 0$ the set $\mathcal{G} = \{B_\varepsilon(x) : x \in M\}$ is an open cover of M, and by (a) there exists a finite cover, which can be expressed as

$$\{B_\varepsilon(x_k) : \{x_k\}_{k=1}^{n} \subset M\}.$$

This is precisely the condition we need for total boundedness.

Likewise, (b) implies that M is complete. This follows from Theorem 7.11, so let's briefly review the relevant parts.

Theorem 7.11

For any metric space (M,d) the following statements are equivalent:

(i) (M,d) is complete

(ii) Let $F_1 \supset F_2 \supset F_3 \supset \cdots$ be a decreasing sequence of nonempty closed sets in M with $\operatorname{diam}(F_n) \to 0$. Then $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

Let $F_1 \supset F_2 \supset \dots$ be any decreasing sequence of nonempty closed sets in M so that $\operatorname{diam}(F_n) \to 0$ as $n \to \infty$.

By the nonempty and containment properties

$$\bigcap_{k=1}^{n} F_k = F_n \neq \emptyset$$

for any finite subcollection $\{F_k\}_{k=1}^{n}$ (possibly relabeled). But then (b) asserts that

$$\bigcap_{k=1}^{\infty} F_k \neq \emptyset$$

which means (ii) holds, and so by (i) M is complete.

We can conclude that if either (α) or (β) holds (and so both hold), M will be both totally bounded and complete, and so will be compact.