Alternative Characterizations of Uniform Continuity

Note Title 8/8/2015

Definition

We say that a function $f:(M,d)\to(N,\rho)$ is uniformly continuous if given any $\varepsilon>0$ there exists $\delta>0$ so that

$$d(x,y)<\delta \implies \rho(f(x),f(y))<\varepsilon$$

$\forall\, x,y \in M$.

We can restate these inequalities as

$$f\left(B_\delta^d(x)\right) \subset B_\varepsilon^\rho(f(x))$$

for all $x \in M$.

We could also use more general sets. I.e., we could say f is uniformly continuous if given any $\varepsilon > 0$ there exists $\delta > 0$ so that for any $A \subset M$

$$\operatorname{diam}_M(A) < \delta \implies \operatorname{diam}_N(f(A)) < \varepsilon.$$

In particular, it's clear from this characterization that a uniformly continuous function maps Cauchy sequences to Cauchy sequences. This is because we can state the Cauchy property as

$$n \geq N \implies \operatorname{diam}\{x_k : k \geq n\} < \varepsilon$$

and we see that

$$\operatorname{diam}_M \{x_k : k \geq n\} < \delta \implies \operatorname{diam}_N \{f(x_k) : k \geq n\} < \varepsilon.$$

We can extend this observation a bit as follows:

Proposition 8.14

If $f : M \to N$ is uniformly continuous, then f maps totally bounded sets into totally bounded sets.

Proof

Let $A \subset M$ be totally bounded, and let $\varepsilon > 0$.

Since f is uniformly continuous, there exists $\delta > 0$ so that $f(B_\delta^d(x)) \subset B_\varepsilon^\rho(f(x)) \ \forall x \in M$.

Since A is totally bounded there exist $\{x_i\}_{i=1}^n \subset M$ so that

$$A \subset \bigcup_{i=1}^n B_\delta^d(x_i).$$

We must have

$$f(A) \subset f\left(\bigcup_{i=1}^n B_\delta^d(x_i)\right) = \bigcup_{i=1}^n f\left(B_\delta^d(x_i)\right)$$

We see that

$$f(A) \subset \bigcup_{i=1}^{n} B_\varepsilon^{\rho}(f(x_i))$$

and this means that $f(A)$ is totally bounded. $\square$