Note Title 8/10/2015

Uniqueness of Completions (a Shorter Proof)

Theorem 7.18 / Corollary 8.17

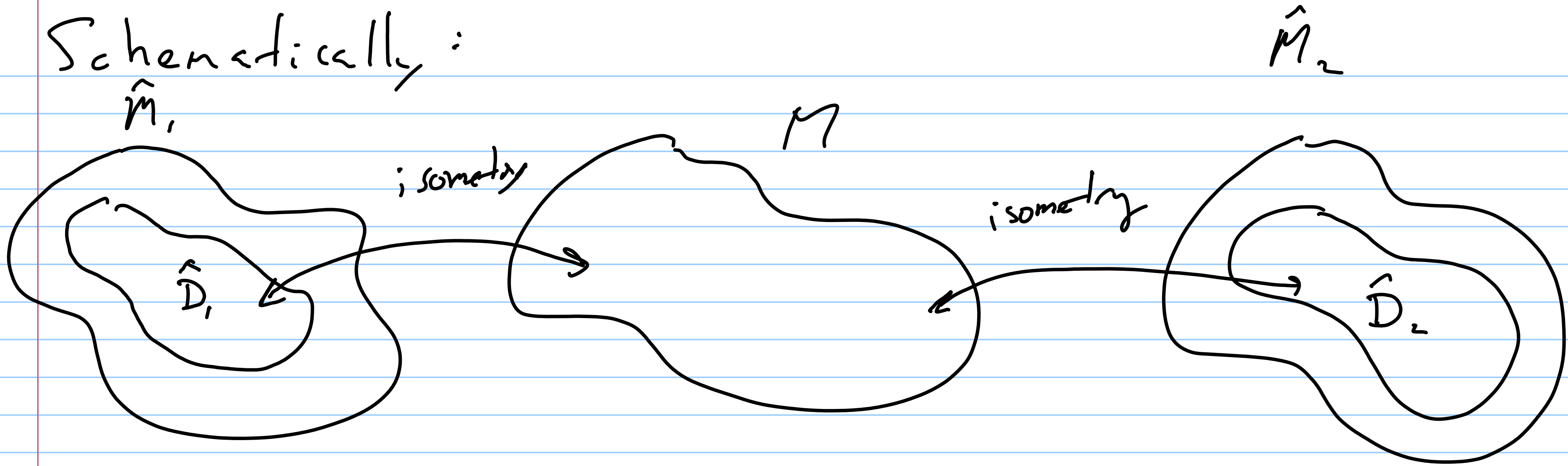
If (M_1, d_1) and (M_2, d_2) are completions of (M, d), then (M_1, d_1) and (M_2, d_2) are isometric.

Second Proof

Notice that if (M, d) is any metric space, and $(\hat{M}_1, \hat{d}_1)$ is any completion of it, then

M is isometric to a dense subset of $\hat{M}$, which we'll denote $\hat{D}$. Suppose $(\hat{M}_2, \hat{d}_2)$ is a second completion of (M,d), so also with a dense subset, in this case $\hat{D}_2$, isometric to M.

Schematically:



These isometries create a map $F : \hat{D}_1 \to \hat{M}_2$, so that $F(\hat{D}_1) = \hat{D}_2$. Now F is an isometry, and so it is uniformly continuous by Problem 8.44. This means we can apply

Theorem 8.16 to extend it to an isometry

$$\hat{F} : \hat{M}_1 \to \hat{M}_2 .$$

There's one bit of fineprint: in Theorem 8.16 the extension $F : M \to N$ is an isometry between M and $f(M)$. (Not necessarily onto N.) We can verify that $\hat{F}$ is onto $\hat{M}_2$ precisely as we did in the proof of Theorem 7.18. $\square$