Properties of Continuous Linear Maps

Theorem 8.20

Let $(V, \|\cdot\|)$ and $(W, |||\cdot|||)$ be normed vector spaces, and let $T: V \to W$ be a linear map. Then the following are equivalent:

(i) T is Lipschitz on V

(ii) T is uniformly continuous on V

(iii) T is continuous on V

(iv) T is continuous at $0 \in V$

(v) There is a constant $C < \infty$ so that

$$|||T(x)||| \leq C \|x\|$$

for $x \in V$.

<u>Proof</u>

First, this is clearly set up so that the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are trivial. We need to show (iv) $\Rightarrow$ (v) and (v) $\Rightarrow$ (i) to close the loop.

Following Carothers, let's start with (v) $\Rightarrow$ (i). If (v) holds, then since $x-y \in V$ we have

$$|||T(x-y)||| \leq C\|x-y\|,$$

which means that T is Lipschitz continuous with Lipschitz constant C.

For (iv) $\Rightarrow$ (v), notice that if T is continuous at 0 then we can find $\delta > 0$ small enough so that

$$\|x\| \leq \delta \implies |||T(x)||| \leq 1$$

(i.e., we take $\varepsilon = 1$, and use $\|x - 0\| = \|x\|$, and $|||T(x) - \underbrace{T(0)}_{0 \text{ by linearity}}||| = |||T(x)|||$).

Now given any $x \neq 0$ (and $x \in V$) consider

$\frac{\delta}{\|x\|} x \in V$, which satisfies

$$\left\| \frac{\delta}{\|x\|} x \right\| = \frac{\delta}{\cancel{\|x\|}} \cancel{\|x\|} = \delta .$$

Also, by linearity of T

$$T\left(\frac{\delta}{\|x\|} x\right) = \frac{\delta}{\|x\|} T(x) .$$

We know

$$\left\| T\left(\frac{\delta}{\|x\|} x\right) \right\| \le 1,$$

so

$$\frac{\delta}{\|x\|} |||T(x)||| \leq 1$$

$$\Rightarrow \quad |||T(x)||| \leq \frac{1}{\delta} \|x\|.$$

But this gives Property (v) with $C = \frac{1}{\delta}$ $\square$