The Space of Bounded Linear Maps

Definition

A linear map $T : (V, \|\cdot\|) \to (W, |||\cdot|||)$ satisfying the condition

$$|||Tx||| \leq C\|x\|$$

for all $x \in V$ is said to be *bounded*.

(Notice that for linear maps, we often write $T(x) = Tx$.)

It's clear that one thing Theorem 8.20 tells us is that a linear map is bounded iff it's continuous.

Notice this is different from our usual notion of boundedness for a real-valued function. For example, $f(x) = x$ on $\mathbb{R}$ is unbounded as a function, but is a bounded linear map because

$$|f(x)| \leq |x|$$

for all $x \in \mathbb{R}$ (so $C = 1$).

In the current setting, a good way to think of a bounded linear map is as a map that takes bounded sets to bounded sets.

Given normed vector spaces $(V, \|\cdot\|)$ and $(W, |||\cdot|||)$, We can consider the space of all bounded linear maps $T: V \to W$, which we denote $B(V, W)$. It's easy to see that $B(V,W)$ is itself a linear vector space: if $S, T \in B(V,W)$ then

$$\text{(i)} \quad S + T \in B(V,W)$$

$$\text{(ii)} \quad \alpha S \in B(V,V)$$

(and the other vector space properties are easily checked).

Let's check (i): We know

$$|||Sx||| \le C_1 \|x\|$$

and

$$|||Tx||| \le C_2 \|x\|,$$

so

$$|||(S+T)x||| = |||Sx + Tx||| \le |||Sx||| + |||Tx|||$$
$$\le C_1 \|x\| + C_2 \|x\| = (C_1 + C_2)\|x\|.$$

In fact $B(V,W)$ is a normal linear space, though to see this we need to define an appropriate norm. Our candidate will be:

$$\|T\| := \inf\{C : |||Tx||| \leq C\|x\| \ \forall x \in V\}.$$

Notice that we could express $\|T\|$ as

$$\|T\| = \inf\left\{C : \frac{|||Tx|||}{\|x\|} \leq C \quad \forall x \in V \setminus \{0\}\right\}$$

So we have

$$\|T\| = \sup_{x \neq 0} \frac{|||Tx|||}{\|x\|}.$$

We'll see in Problem 8.82 that this is also equivalent to

$$\|T\| = \sup_{\substack{\|x\| \leq 1 \\ x \neq 0}} |||Tx|||$$

and

$$\|T\| = \sup_{\|x\|=1} |||Tx|||$$

Notice that

$$\frac{|||Tx|||}{\|x\|} \le \|T\|$$

for all $x \neq 0$, so we have the Cauchy-Schwarz type relation

$$|||Tx||| \le \|T\| \, \|x\| .$$

One final comment: we'll see in Problem 8.84 that if W is a Banach space then so is $B(V, W)$, even if V is only a normed linear space.