Note Title 8/12/2015

Equivalence of Norms on a Single Vector Space

Corollary 8.21

Let $\|\cdot\|$ and $|||\cdot|||$ be two norms on a single vector space V. Then $\|\cdot\|$ and $|||\cdot|||$ are equivalent iff there are constants $0 < c, C < \infty$ so that

$$c\|x\| \leq |||x||| \leq C\|x\|$$

$\forall x \in V$. I.e., equivalence is the same as strong equivalence.

<u>Proof</u>

For ($\Leftarrow$) strong equivalence always implies equivalence.

For ($\Rightarrow$), by definition $\|\cdot\|$ are $|||\cdot|||$ are equivalent if

$$i : (V, \|\cdot\|) \longrightarrow (V, |||\cdot|||)$$

and

$$i^{-1} : (V, |||\cdot|||) \longrightarrow (V, \|\cdot\|)$$

are both continuous. Since the identity map is linear, this means (by Theorem 8.20) that i and i^{-1} are both bounded.

To say that i is bounded means there exists some constant C_1 so that

$$|||x||| \leq C_1 \|x\|$$

for all $x \in V$, and to say i^{-1} is bounded means that there exists some constant C_2 so that

$$\|x\| \leq C_2 |||x|||.$$

Combining these, we have:

$$\frac{1}{C_2}\|x\| \leq |||x||| \leq C_1 \|x\|$$

This gives the claim with $c = \frac{1}{C_2}$ and $C = C_2$.

$\square$