The Oscillation of a Function on an Interval

Note Title 8/15/2015

So far we've thought mainly about continuous functions. In this section, we'll think about discontinuous functions, focusing on $f:\mathbb{R}\to\mathbb{R}$. We'll be interested in the set of points where f is discontinuous, and we'll denote this set $D(f)$.

Examples

1. Recall from Theorem 2.17 that if $f : (0,1) \to \mathbb{R}$ is monotone then f can have at most a countable number of discontinuities. So $D(f)$ is countable.

2. For $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & x \text{ rational} \\ 1 & x \text{ irrational}, \end{cases}$$

we have

$$D(f) = [0,1]$$

(See Problem 1.47.)

3. For $f : [0,1] \longrightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & x \in \Delta \\ 1 & x \notin \Delta \end{cases}$$

we have $D(f) = \Delta$. (Carothers states this in different ways in Problems 2.25, 5.4, and 9.6.)

We can characterize $D(f)$ as follows:

We have $a \in D(f)$ iff there exists $\varepsilon > 0$ so that given any $\delta > 0$ we have

$$|f(x) - f(a)| \geq \varepsilon$$

for some x with $|x-a| < \delta$.

We can express this in the following way: given any open bounded interval I containing a, we have

$$\sup_{x,y \in I} |f(x) - f(y)| \geq \varepsilon .$$

First, by taking $y = a$ we see that if

$$|f(x) - f(a)| \geq \varepsilon$$

then

$$\sup_{x,y \in I} |f(x) - f(y)| \geq \varepsilon$$

On the other hand, suppose the supremum condition holds but the previous condition does not. Take $\delta > 0$ small enough so that

$$|x - a| < \delta \implies |f(x) - f(a)| < \frac{\varepsilon}{4}$$

Then

$$|f(x) - f(y)| = |f(x) - f(a) + f(a) - f(y)|$$

$$\leq |f(x) - f(a)| + |f(a) - f(y)|$$

$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$$

$\forall\ x, y \in I = (a - \delta, a + \delta)$, and this is a contradiction to the supremum condition.

Recall that by the definition of diameter

$$\operatorname{diam} f(I) = \sup_{x,y \in I} |f(x) - f(y)|$$

so our supremum condition asserts

$$\operatorname{diam} f(I) \geq \varepsilon$$

for any bounded open interval I containing a.

Definition

Given a bounded interval I, we define $\omega(f;I)$, the oscillation of f on I, by

$$\omega(f;I) = \sup_{x,y\in I} |f(x)-f(y)| = \operatorname{diam} f(I).$$

Clearly, $0 \le \underset{\text{on } I}{\omega(f;I)} \le 2 \sup_{x\in I} |f(x)|$. If f is unbounded, $\omega(f;I) = \infty$.