The Complement of a First Category Set is Dense

Corollary 9.12

In a complete metric space (M, d), the complement of any first category set is nonempty. In fact, it is dense.

Proof

Let A be a first category set in M, so by definition

$$A = \bigcup_{n=1}^{\infty} E_n$$

for a sequence (E_n) of nowhere dense sets. By De Morgan's Laws, we have

$$A^c = \bigcap_{n=1}^{\infty} E_n^c .$$

Notice that

$$\bigcap_{n=1}^{\infty} \overline{E_n}^c \subset \bigcap_{n=1}^{\infty} E_n^c = A^c .$$

But we know that if E_n is nowhere dense in M

then $\bar{E}_n^c$ is dense in M, so by the Baire Category Theorem $\bigcap_{n=1}^{\infty} \bar{E}_n^c$ is dense in M.

But $\bigcap_{n=1}^{\infty} \bar{E}_n^c$ is a subset of A^c, so certainly A^c is dense in M. $\square$