Note Title

# Some Consequences of the Least Upper Bound Axiom

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## Lemma 1.2

If $x$ and $y$ are positive real numbers, then there is some positive integer $n$ so that $nx > y$.

## Proof

We proceed by contradiction: suppose that no such $n$ exists, which means

$$nx \leq y$$

for all $n \in \mathbb{N} = \{1, 2, \ldots\}$. This means that the set

$$A = \{nx : n \in \mathbb{N}\}$$

is bounded above by $y$, and so

$$s = \sup A$$

is finite.

Now since $s - x < s$ there must be some element of $A$ in the gap

$$s - x < a \leq s .$$

(or $s - x$ would be $\sup A$). We can write $a = nx$ for some $n \in \mathbb{N}$, and this implies

$$s - x < nx \leq s .$$

We see that $s < (n+1)x$. But $(n+1)x \in A$, and this contradicts the existence of $s$, which in turn contradicts our assumption. $\square$

## Theorem 1.3

If $a$ and $b$ are real numbers with $a < b$, then there is a rational number $r \in \mathbb{Q}$ with $a < r < b$.

## Proof

Since $b - a > 0$ we know from Lemma 1.2 that there exists a positive integer $I$ so that

$$q(b-a) > 1.$$

(I.e., $x$ and $y$ from Lemma 1.2 are replaced with $b-a$ and $1$ here.) Since $qb - qa > 1$ there must be an integer between $qa$ and $qb$. I.e., there is an integer $p$ so that

$$qa < p < qb$$

But then $a < \frac{p}{q} < b$, and we have the claim for $r = p/q$. $\square$