Monotone Convergence Theorem

Note Title 6/8/2015

Sequences

We'll denote a sequence of numbers

$$x_1, x_2, \ldots$$

by (x_n). So, for example, $\left(\frac{1}{n}\right)$ would indicate $1, \frac{1}{2}, \frac{1}{3}, \ldots$

We say x_n converges to x, written $x_n \longrightarrow x$, if given any $\varepsilon > 0$ there exists an integer N so that

$$n \geq N \implies |x - x_n| < \varepsilon.$$

Definition

We say that a sequence (x_n) is increasing if $x_{n+1} \geq x_n$ $\forall\, n \in \mathbb{N}$, and decreasing if $x_{n+1} \leq x_n$ $\forall\, n \in \mathbb{N}$. We say a sequence is monotone if it is either increasing or decreasing.

Notice that when we say (x_n) is increasing we really mean non-decreasing, and when we say (x_n) is decreasing we really mean non-increasing.

We say (x_n) is <u>strictly increasing</u> if $x_{n+1} > x_n$ $\forall\, n \in \mathbb{N}$, and similarly for <u>strictly decreasing</u>.

Theorem 1.4 (Monotone Convergence Theorem)

A monotone, bounded sequence of real numbers converges.

Proof

Let $(x_n) \subset \mathbb{R}$ be monotone and bounded. First, suppose (x_n) is increasing, and set $x = \sup_n (x_n)$, which we know exists.

Let $\varepsilon > 0$. Since $x - \varepsilon < x = \sup_n (x_n)$
we must have $x_N > x - \varepsilon$ for some integer
N. But by monotonicity this means

$$n \geq N \implies x - \varepsilon < x_N \leq x_n.$$

But this precisely says

$$n \geq N \implies |x - x_n| < \varepsilon$$

(here, $x - x_n \geq 0$, so $x - x_n = |x - x_n|$).

This is the definition of convergence, so we have the claim for increasing sequences.

On the other hand, if (x_n) is decreasing, then $(-x_n)$ is increasing, and we know from the first part that $-x_n$ converges to $-x$.
So x_n converges to $\inf_n (x_n)$. $\square$