Note Title 6/8/2015

p-adic expansions

Proposition 1.8

Let p be an integer, $p \geq 2$, and let $0 \leq x \leq 1$. Then there is a sequence of integers (a_n) with $0 \leq a_n \leq p-1$ for all n such that

$$x = \sum_{n=1}^{\infty} \frac{a_n}{p^n}.$$

Proof

If $x = 0$, we simply take $a_n = 0$ $\forall n$, so we can focus on $x \in (0,1]$.

In this case, we'll construct the sequence inductively, beginning with a_1, which we take to be the largest integer satisfying $\frac{a_1}{p} < x$. Since $x \leq 1$, we must have $a_1 < p$, and since a_1 is an integer this means $a_1 \le p-1$. Also, since a_1 is the largest integer for which we have $\frac{a_1}{p} < x$, we must have

$$\frac{a_1}{p} < x \le \frac{a_1+1}{p}.$$

Next, choose a_2 to be the largest integer satisfying

$$\frac{a_1}{p} + \frac{a_2}{p^2} < x,$$

so that as above

$$\frac{a_1}{p} + \frac{a_2}{p^2} < x \le \frac{a_1}{p} + \frac{a_2+1}{p^2}$$

$$= \frac{a_1}{p} + \frac{a_2}{p^2} + \frac{1}{p^2}$$

Continuing in this way, we construct a sequence so that

$$\sum_{k=1}^{n} \frac{a_k}{p^k} < x \le \sum_{k=1}^{n} \frac{a_k}{p^k} + \frac{1}{p^n}$$

Since $p \geq 2$, when we take $n \to \infty$ we get

$$\sum_{k=1}^{\infty} \frac{a_k}{p^k} = x.$$

$\square$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{p^n}$ is called a base p (or p-adic) decimal expansion for x.

We're most familiar with the case $p = 10$, for which $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{10^n}$ is the usual decimal expansion $.a_1 a_2 a_3 \ldots$

For general integer $p \geq 2$ we sometimes write

$$\sum_{n=1}^{\infty} \frac{a_n}{p^n} = .a_1 a_2 a_3 \ldots \quad (\text{base } p)$$

We can use this to approximate real numbers by sequences of rational numbers, and we can use this to define properties of irrational numbers.

For example, if $a > 0$ is a base, and n is an integer, we define

$$a^n = \underbrace{a \cdot a \cdots a}_{n \text{ times}}$$

Likewise, we define $a^{1/m}$ as the value so that

$$\underbrace{a^{1/m} \cdot a^{1/m} \cdots a^{1/m}}_{m \text{ times}} = a.$$

We can now make sense of

$$a^{\frac{n}{m}} = \left(a^n\right)^{1/m} = \left(a^{1/m}\right)^n$$

But what about a^x, if x is irrational?

We can define

$$a^x \underbrace{:=}_{\text{define}} \sup\{ a^r : r \in \mathbb{Q},\ r < x \}.$$